DISCONTINUOUS FINITE VOLUME METHODS FOR OPTIMAL CONTROL PROBLEMS

A thesis submitted
in partial fulfillment for the degree of

Doctor of Philosophy

by

RUCHI SANDILYA

Department of Mathematics
INDIAN INSTITUTE OF SPACE SCIENCE AND TECHNOLOGY
Thiruvananthapuram - 695 547

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CERTIFICATE

This is to certify that the thesis entitled **Discontinuous finite volume methods for optimal control problems** submitted by **Ruchi Sandilya** to the Indian Institute of Space Science and Technology, Thiruvananthapuram, in partial fulfillment for the award of the degree of **Doctor of Philosophy** is a *bona fide* record of research work carried out by her under my supervision. The contents of this thesis, in full or in parts, have not been submitted to any other Institution or University for the award of any degree or diploma.

Dr. Sarvesh Kumar
Supervisor
Department of Mathematics

Thiruvananthapuram
August 2016

Counter signature of HOD with seal

DECLARATION

I declare that this thesis entitled **Discontinuous finite volume methods for optimal control problems** submitted in partial fulfillment of the Degree of Doctor of Philosophy is a record of original work carried out by me under the supervision of **Dr. Sarvesh Kumar**, and has not formed the basis for the award of any other degree or diploma, in this or any other Institution or University. In keeping with the ethical practice in reporting scientific information, due acknowledgements have been made wherever the findings of others have been cited.

Thiruvananthapuram-695547 Ruchi Sandilya

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ABSTRACT

The main objective of this thesis is to develop and analyze discontinuous finite volume methods for the approximation of distributed optimal control problems governed by certain partial differential equations subject to pointwise control constraints. In view of applications, we consider optimal control problems governed by semilinear elliptic, parabolic and hyperbolic problems, and Brinkman equations (that describe flow of an incompressible viscous fluid through a porous medium). For the discretization of state and costate variables, we utilize piecewise linear discontinuous finite volume schemes, whereas three different strategies are used for control approximation: variational discretization approach—in which control set is not discretized explicitly but discretized by a projection of the discrete costate variables, as well as piecewise constant and piecewise linear discretizations. As the resulting discrete optimal systems are non-symmetric, we employ the so-called *optimize-then-discretize* approach to approximate the control problem. *A priori* error estimates in suitable natural norms are derived for control, state and costate variables. Further, numerical experiments are presented to illustrate the performance of the proposed schemes and to confirm the predicted accuracy of the theoretical convergence rates. Finally, based on theoretical and computational observations, this thesis addresses concluding remarks and future work regarding possible extensions of present work to more application based and real life problems.

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CHAPTER 1

Introduction

The main focus of this thesis is on the development of accurate and robust numerical schemes for the discretization of optimal control problems governed by semilinear elliptic, parabolic and hyperbolic equations, and also by Brinkman equations which have significant applications in the field of science and engineering. Here we consider discontinuous finite volume methods that, by construction preserve all desirable features of discontinuous Galerkin and finite volume methods .

1.1 Motivation

The theory of optimal control problems governed by partial differential equations was introduced by Lions in [54]. In optimal control problems, the general idea is to vary an input quantity (called control) in such a way that the output quantity (called state) minimizes the objective functional. The input can be a function prescribed on the boundary (called boundary control) or distributed all over the domain (called distributed control), and the output is the solution of the partial differential equation. Due to physical and technical limitations, one needs to impose some restrictions on control and/or state. In our case, we consider control and state coupled by partial differential equations with constraints only on the control. Such optimization problems can be abstractly written as

$$\min_{(y,u)\in\mathcal{Y}\times\mathcal{U}} J(y,u) \quad \text{subject to} \quad e(y,u)=0,\ u\in U_{ad},$$

where $\mathcal{Y}$ and $\mathcal{U}$ are the respective Banach spaces for the state variable y and control variable u. The term $J(y,u)$ represents the objective or cost functional, $e(y,u)=0$ denotes a PDE and $U_{ad}\subset\mathcal{U}$ is a closed convex set representing the control constraints.

The mathematical theory of optimal control problems has developed rapidly as an active area of research in the field of applied mathematics. In general, it is difficult to obtain the analytical solution of optimal control problems and hence one has to rely on some accurate and robust numerical techniques to compute approximate solutions. For the numerical treatment of optimal control problems the field of mathematics is well established. For the relevant literature, we refer to [8, 33, 43, 45, 74] and references therein. The gradient methods were among the first techniques to solve optimal control problems governed by partial differential equations. Although these methods converges slowly, but it can be implemented easily and therefore, are well suited for numerical tests of complex and nonlinear problems. The expositions of projection gradient method and its convergence properties can be found in [43]. Another efficient and commonly used numerical method is primal-dual active set strategy (proposed in [8]). In this method, at each iteration step one updates active sets for the upper and lower box constraints and the control is fixed in the next step by taking the corresponding upper and lower threshold value. The overall idea is to approximate the constrained optimal control problem by a sequence of unconstrained problems, using active sets. This method can be interpreted as semi-smooth Newton method and converges superlinearly. There has also been significant recent interest in preconditioning and iterative solvers for PDE-constrained optimal control problems, for instance see [73, 74].

With the advancement of numerical techniques and faster computational facilities in past few decades, this area has increased its applications in the industrial, medical and economic sectors. With the growing popularity of optimal control problems, the researchers are developing more advanced techniques for the accurate simulation of these problems. It is known that PDEs are used to model many physical processes like heat conduction, diffusion, electromagnetic waves, fluid flows, freezing processes and many more, the optimization problems governed by partial differential equations are very crucial to engineering applications. In particular, semilinear elliptic, parabolic as well as hyperbolic optimal control problems are used to describe many real world phenomena such as heating processes, noise suppression, laser hardening, welding of steel, laser thermotherapy (used for cancer treatment) etc. Also, the Brinkman equations describing the motion of an incompressible viscous fluid within an array of porous particles. Therefore, in this study we will pay our attention on numerical solutions of these optimal control problems by using an advance method so-called discontinuous finite volume methods.

1.1.1 Discontinuous finite volume methods

Contrary to the conforming and non-conforming finite element (FE) methods, for the case of discontinuous Galerkin (DG) methods, the inter-element continuity criteria is not imposed on finite dimensional spaces which makes DG methods more suitable for obtaining a high order of accuracy, high parallelizability and localizability, and easy handling of complicated geometries. Other good features of DG methods include local mesh adaptivity, element-wise conservative, allow different degree polynomials in different elements and easily handle the boundary conditions. These properties have made these methods very appealing to the scientific community and developed rich literature concerning their numerical analysis and applications for many types of PDEs (see [4, 5, 70, 75] and the references therein). On the other hand, the finite volume element (FVE) methods can be considered as Petrov-Galerkin methods in which the finite dimensional trial and test spaces are chosen as continuous piecewise linear polynomials on the finite dimensional partition of the domain and piecewise constant functions over the control volumes, respectively (see the early work [14, 28, 44] and the recent review [53]). Due to local conservation properties and other attractive features, FVE methods are widely used in computational fluid dynamics (cf. [27, 31, 51, 72]), for more applications and details of these methods we refer to [64]. Since the test space associated with the dual grid is piecewise constant, FVE methods are computationally less expensive than standard FE methods and still achieve the same convergence rates. The disadvantage is that the low regularity in the test function demands extra regularity on the exact solution or the given data in order to achieve optimal L^2 error estimates. For instance, for non-homogeneous elliptic problems, to derive optimal L^2 error estimates, one requires either an exact solution in H^3 or a source term globally in H^1 (see [35]).

In order to utilize the desirable features of both FVE and DG methods, a hybrid scheme called discontinuous finite volume (DFV) methods was proposed in [81], and unified analysis for elliptic problems was presented in [29]. In DFV methods, discontinuous piecewise linear functions conform the trial space, whereas piecewise constant test functions are used in a finite volume fashion. An advantage of these schemes over standard FVEs is that the control volumes have support only inside the triangle in which they belong and there is no contribution from the adjacent triangles which is different from the case of conforming FVM. This makes DFV methods more suitable for parallel computing. An adaptive DFV method for elliptic problems was developed and analyzed in [55]. Later, with the appropriate modifications, these methods were applied to elliptic, parabolic and certain fluid flow problems (for details, see [9, 12, 13, 49, 50, 52, 56, 82, 83, 84]). In view of these desirable properties, in this thesis, we use DFV methods and extend the available results to cover the case of control problems as the ones mentioned above.

1.2 Related work and specific contributions

Usually, the treatment of semilinear partial differential equations is considerably more difficult and the theory of their optimal control is a delicate issue and requires some additional knowledge. In contrast to the linear case, the optimal control problems governed by semilinear state equations are nonconvex, even if the cost functional is convex and hence may exhibit multiple solutions. In order to deal with the difficulty in deriving the convergence results, the notion of local optimal controls is introduced in [16] and the error estimates for a fixed local optimal control of semilinear elliptic control problem is derived. The involved spaces are chosen appropriately to guarantee the differentiability of control-to-state operator and from which the convergence follows.

Classical methods like FE schemes have been widely applied to solve such optimal control problems; error estimates for FE discretizations and computations of optimal control problems governed by linear and semilinear elliptic and parabolic optimal control problems can be found in [16, 17, 61, 62, 67]. For hyperbolic control problems some *a priori* and *a posteriori* error estimates as well as adaptive FE computations were obtained in ([46, 47, 65]). The theoretical aspects of fluid control problems can be found in the classical works [1, 54], whereas their numerical solution associated to FE methods has a rich literature (see e.g. [11, 34, 38, 69, 74, 78] and the references therein). Most of these contributions employ conforming piecewise linear FE discretizations for state and costate variables and the control variable is discretized using piecewise constant or linear polynomials. It has been found that the convergence rate for control discretization is of $\mathcal{O}(h)$ and $\mathcal{O}(h^{3/2})$ when piecewise constant and linear polynomials are used, respectively. On the other hand, in the variational discretization approach for optimal control problems (for details see [42]), if the control constraints in which control set is not discretized explicitly but discretized by a projection of the discrete costate variables, an improved convergence rate of $\mathcal{O}(h^2)$ can be achieved for control variables. However in this approach the discrete control variable does not belong to the FE space associated with the given mesh and therefore one has to deal with a nonstandard implementation and more involved stopping criteria for numerical algorithm. A similar convergence result holds if using graded meshes instead of uniform partitions [68]. Also, in [63] a piecewise constant discretization is utilized, and $\mathcal{O}(h^2)$-convergence is achieved by using a postprocessing step based on a projection formula. This technique was later extended to optimal control problems governed by parabolic and Stokes equations in [62, 78].

A few contributions are available (cf. [25, 61, 62, 67]) which deal with DG methods for optimal control problems constrained to linear and semilinear parabolic equations. Furthermore, a few results are also available on DG methods applied to flow control problems (see e.g., [18, 19, 26]). However, one of the drawbacks of DG methods is that the larger number of degrees of freedom leads to high computational cost to achieve a fixed accuracy. Moreover, motivated by the computational efficiency and simplicity, FVE methods have been used for the approximation of linear elliptic, parabolic and hyperbolic optimal control problems (see e.g. [58, 59, 60]) and *a priori* error estimates have also been established. In these articles, variational discretization approach is used to approximate control variable and optimal order of convergence is obtained.

In this thesis we are interested in the analysis of DFV methods which would provide accurate and robust numerical solution of optimal control problems governed by elliptic, parabolic, hyperbolic and Brinkman equations. Also, an attempt has been made to derive optimal *a priori* error estimates for state, costate and control variables in suitable natural norms and some numerical examples are considered to test the performance of the scheme. We stress that to the best of our knowledge, DFV methods are not discussed in literature as far as numerical approximation of optimal control problems is concerned. Further, three different techniques have been used for the discretization of the control variable: variational discretization, piecewise constant and piecewise linear discretizations.

For nonlinear or semilinear problems, standard methods like FE and FVE methods lead to large nonlinear algebraic systems (obtained by discretizating the governing nonlinear equations), which are then solved by Newton iterations. At each linearization step, one needs to compute the Jacobian matrix, involving typically complex derivatives, that may be very time-consuming and at times not even available. In order to overcome this difficulty, the idea of interpolated coefficients was introduced and analyzed by Zlamal in [88] for approximating semilinear parabolic problems and for elliptic problems in [85] in context of FE methods. With the introduction of interpolated coefficients, it was observed that the computation cost is reduced greatly as the Jacobian matrix can be computed in a simple way as the derivative of nonlinear term involves direct multiplication with mass matrix and Jacobian matrix is updated once in each iteration of Newton method. Motivated by the computational advantage of interpolated coefficient method, we have extended this idea to cover the context of DFV methods for semilinear optimal control problems.

For the numerical solution of optimal control problems, two main approaches are available from the literature: the *optimize-then-discretize* and the *discretize-then-optimize* methods. In the first case, optimality conditions at the continuous level are formulated first and then one proceeds to the discretization step; whereas in *discretize-then-optimize* approach one first discretizes the continuous problem and then derives the optimality conditions accordingly. For non-symmetric discrete formulations, these two approaches need not coincide as they may lead to different discrete adjoint equations (see [11]). In general, finite volume element formulation is non-symmetric and the authors in [58, 59, 60] have employed *optimize-then-discretize* technique to discretize the optimal control problems. In [30] the effect of SUPG finite element method on the discretization of optimal control problems governed by the linear advection-diffusion equation was studied and the authors observed that the *optimize-then-discretize* approach leads to asymptotically better approximate solutions than *discretize-then-optimize* approach. Also, it is clearly mentioned in [58] that out of these two, the preference will be given to *optimize-then-discretize* approach, because in this approach one can use the same discretization for state and costate equations and then use well established results for the accomplishment of error analysis. In the light of these articles and applicability of *optimize-then-discretize* approach for non-symmetric formulation, in this thesis, we will also undertake the same strategy (*optimize-then-discretize*) for the approximation of the concerned optimal control problem.

1.3 Preliminaries

In this Section, we introduce some standard notations and basic notions from functional analysis to be used throughout the thesis.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded, convex polygonal domain with Lipschitz boundary $\partial\Omega$. For $p \in [1, \infty)$, let $L^p(\Omega)$ denote the linear space of all (equivalence classes of) Lebesgue measurable functions ϕ, defined on Ω, that satisfy

$$\int_\Omega |\phi(x)|^p dx < \infty.$$

In this connection, the functions are considered to belong to the same equivalence class if they differ only on a set of measure zero. The space $L^p(\Omega)$, with $1 < p < \infty$, and equipped with the norm

$$\|\phi\|_{L^p(\Omega)} := \left(\int_\Omega |\phi(x)|^p dx \right)^{1/p}$$

is a Banach space. The space $L^\infty(\Omega)$ is the Banach space of all (equivalence classes of) Lebesgue measurable and essentially bounded functions, endowed with the norm

$$\|\phi\|_{L^\infty(\Omega)} := ess \sup_{x \in \Omega} |\phi(x)|.$$

It is well known that $L^2(\Omega)$ is a Hilbert space with respect to the inner product $(\cdot, \cdot)$ defined by

$$(\phi, \psi) := \int_\Omega \phi(x)\psi(x)dx.$$

For $s \in \mathbb{N}$ and $p \in [1, \infty]$, the classical Sobolev space $W^{s,p}(\Omega)$ is defined as the linear space of all functions $\phi \in L^p(\Omega)$ having distributional derivatives $D^\alpha\phi \in L^p(\Omega)$ for all multi-indices α of order $|\alpha| \leq s$, and is equipped with the norm

$$\|\phi\|_{W^{s,p}(\Omega)} = \|\phi\|_{s,p,\Omega} := \left(\sum_{|\alpha| \leq s} \int_\Omega |D^\alpha \phi(x)|^p dx \right)^{1/p}.$$

Furthermore, we introduce the semi-norm

$$|\phi|_{s,p,\Omega} := \left(\sum_{|\alpha|=s} \int_{\Omega} |D^{\alpha}\phi(x)|^{p} dx \right)^{1/p}.$$

Analogously, for $p = \infty$,

$$\|\phi\|_{s,\infty,\Omega} := \max_{|\alpha| \leq s} \|D^{\alpha}\phi\|_{L^{\infty}(\Omega)}.$$

The spaces $W^{s,p}(\Omega)$ are Banach spaces. For simplicity, we use the abbreviation $H^{s}(\Omega) := W^{s,2}(\Omega)$ and define $W^{0,p}(\Omega) := L^{p}(\Omega)$. We note that $H^{s}(\Omega)$ is a Hilbert space with respect to the inner product

$$(\phi, \psi)_{s,\Omega} := \sum_{|\alpha| \leq s} \int_{\Omega} D^{\alpha}\phi(x) D^{\alpha}\psi(x) dx, \quad \forall \phi, \psi \in H^{s}(\Omega)$$

and the induced norm

$$\|\phi\|_{s,\Omega} := \left(\sum_{|\alpha| \leq s} \int_{\Omega} |D^{\alpha}\phi(x)|^{2} dx \right)^{1/2}.$$

The space $H_0^1(\Omega)$ is characterized by

$$H_0^1(\Omega) := \{\phi \in H^1(\Omega) : \phi = 0 \text{ on } \partial\Omega\}.$$

Let T be a positive time that defines the time interval $I := (0, T)$, then for $p \in [1, \infty)$ and s a non-negative integer, we denote by $L^{p}(H^{s})$, the Banach (or Bochner-type) space of all L^{p} integrable vector valued functions $\phi(t) : I \longrightarrow H^{s}(\Omega)$ with the norm given by

$$\|\phi\|_{L^{p}(H^{s})} := \left(\int_{0}^{T} \|\phi(t)\|_{s,\Omega}^{p} dt \right)^{1/p}.$$

Analogously, $L^{\infty}(H^{s})$ is the Banach space of all essentially bounded vector valued functions $\phi(t) : I \longrightarrow H^{s}(\Omega)$ endowed with the norm

$$\|\phi\|_{L^{\infty}(H^{s})} := ess \sup_{t \in I} \|\phi(t)\|_{s,\Omega}.$$

We will also frequently use the following standard inequalities.

- **Young's inequality.** If a and b are non-negative real numbers, then for every $\varepsilon > 0$, the following inequality holds

$$ab \le \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}.$$

- **Cauchy-Schwarz inequality.** If $\{a_i\}_{i=1}^N$ and $\{b_i\}_{i=1}^N$ are non-negative real numbers. Then

$$\left(\sum_{i=1}^{N} a_i b_i\right) \le \left(\sum_{i=1}^{N} a_i^2\right)^{1/2} \left(\sum_{i=1}^{N} b_i^2\right)^{1/2}.$$

- **Cauchy-Schwarz inequality for integrals.** Let $\phi, \psi \in L^2(\Omega)$. Then

$$\left|\int_\Omega \phi(x)\psi(x)dx\right| \le \left(\int_\Omega |\phi(x)|^2 dx\right)^{1/2} \left(\int_\Omega |\psi(x)|^2 dx\right)^{1/2}.$$

- **Poincaré inequality.** Let $\Omega \subset \mathbb{R}^d$ be a bounded open subset. Then there exists a positive constant $C = C(\Omega)$, such that

$$\|v\|_{0,\Omega} \le C|v|_{1,\Omega}, \quad \forall v \in H_0^1(\Omega).$$

- **Gronwall's inequality.** Let $g(t)$ and $h(t)$ be continuous functions on interval $t_0 \le t \le t_0 + a$ with $h(t) \ge 0$. If a continuous function $\phi(t)$ has the following property

$$\phi(t) \le g(t) + \int_{t_0}^{t} \phi(s)h(s)ds, \quad t_0 \le t \le t_0 + a,$$

then

$$\phi(t) \le g(t) + \int_{t_0}^{t} g(s)h(s)\text{exp}\left(\int_{s}^{t} h(\tau)d\tau\right)ds, \quad t_0 \le t \le t_0 + a,$$

In particular, when $g(t) = C$ is a non-negative constant, then we have

$$\phi(t) \leq C\text{exp}\left(\int\limits_{t_0}^{t} h(s)ds\right), \quad t_0 \leq t \leq t_0 + a.$$

- **Discrete Gronwall's inequality.** Let $\{\xi^n\}$ be a sequence of non-negative numbers satisfying

$$\xi^n \leq \alpha^n + \sum_{j=0}^{n-1} \beta^j \xi^j, \quad n \geq 0,$$

where $\{\alpha^n\}$ is a non-decreasing sequence and $\beta^j \geq 0$. Then

$$\xi^n \leq \alpha^n \text{exp}\left(\sum_{j=0}^{n-1} \beta^j\right), \quad n \geq 0.$$

In addition, we state the definitions of the derivatives which will be helpful in formulating the optimality conditions in later Chapters of this thesis. In what follows, we suppose X, Y be two normed spaces, X_0 be a non-empty subset of X and $g : X_0 \longrightarrow Y$ be a given map.

Definition 1.3.1. ***(Directional derivative)*** *Let $x \in X_0$ and $v \in Y$ be given. If the limit*

$$g'(x)(v) := \lim_{\nu \to 0} \frac{g(x + \nu v) - g(x)}{\nu}$$

exists, then it is called directional derivative of g at x in the direction v.

Definition 1.3.2. ***(Gâteaux derivative)*** *The mapping g is called Gâteaux differentiable at $x \in X_0$, if the directional derivative $g'(x)$ is a continuous linear mapping from X to Y. Then $g'(x)$ is referred to as Gâteaux derivative of g at x.*

Definition 1.3.3. ***(Fréchet derivative)*** *The mapping g is called Fréchet differentiable at $x \in X_0$, if the continuous linear mapping $g'(x) : X \longrightarrow Y$ satisfies the following property*

$$\lim_{\|v\|_X \to 0} \frac{\|g(x + v) - g(x) - g'(x)v\|_Y}{\|v\|_X} = 0.$$

The operator $g'(x)$ is then called Fréchet derivative of g at x.

Definition 1.3.4. ***(Convex functional)*** *Let $S \subset \mathbb{R}^d, d = 2, 3$ be a non-empty convex set. Then a functional $f : S \longrightarrow \mathbb{R}$ is said to be convex if*

$$f(\nu x_1 + (1-\nu)x_2) \leq \nu f(x_1) + (1-\nu)f(x_2),\ \forall \nu \in [0,1]\, \textit{and}\, x_1, x_2 \in S.$$

The functional f is said to be strictly convex if above condition holds with strict inequality whenever $x_1 \neq x_2$ and $\nu \in (0, 1)$.

Remark 1.3.5. *Throughout this thesis, the notation C is used to denote a generic positive constant which may take different values at different places.*

1.4 Outline

This thesis is organized as follows. Chapter 1 is introductory in nature and applicability of optimal control problems is discussed. This Chapter also recalls the advancement of DFV methods and recent developments of numerical schemes for the approximation of control problems.

In Chapter 2, we analyze the convergence of the proposed scheme applied to distributed optimal control problems governed by a class of second order semilinear elliptic equations. For smooth and clear presentation, we have divided this Chapter into two parts. The first part of Chapter 2 deals with the theoretical development and discrete formulation of the proposed DFV schemes for linear elliptic optimal control problems, and in the second part, we extend this analysis to semilinear elliptic optimal control problems.

Chapters 3 and 4 are devoted for the study of DFV approximations of semilinear parabolic and hyperbolic optimal control problems by following the analysis of semilinear elliptic case. The spatial discretization of state and costate variables follows DFV schemes with element-wise linear functions and the time discretization is based on implicit finite difference schemes.

In Chapter 5, we extend the analysis of DFV discretization of linear elliptic optimal control problem (carried out in Chapter 2) to optimal control problem governed by Brinkman equations written in terms of velocity and pressure. For the discretization of state and costate velocities and pressure, a lowest order DFV scheme is used. Moreover, in our numerical experiments we have compared the performance of proposed method with other classical methods.

In each of these Chapters, we have employed three different methodologies for control discretization: variational discretization, piecewise constant and linear discretization. *A priori* error estimates (for these three approaches) in suitable norms are derived for all the variables. Moreover, numerical experiments are presented to validate theoretical findings and to judge the performance of the method. For our numerical implementation, we have utilized the idea of interpolated coefficients for the approximation of semilinear problems, as it greatly reduces the computational cost.

Finally, Chapter 6 is devoted to the critical assessments of the present work, and also highlighted the theoretical findings of each Chapters. Also, conclusions have been drawn in view of the theoretical and computational observations. We conclude this Chapter with possible extension of the present work to more applicable problems, for instance, convection dominated diffusion problems, transport problems and boundary control problems.

CHAPTER 2

Semilinear elliptic optimal control problems

The aim of this Chapter is to study discontinuous finite volume (DFV) approximations for optimal control problems governed by semilinear elliptic problems. Many physical quantities like electric potential, a stationary temperature distribution, a scattered field, or a velocity potential are represented by elliptic equations. Optimal control problems governed by semilinear partial differential equations (introduced in [54]) have a number of applications in the field of physics, medicine and engineering, for instance, notable examples include, the optimal control of current in a cathodic protection system, the optimal control problem in radiation and scattering, the defibrillation procedures in cardiac electrophysiology and many others.

Although the main focus of this Chapter is to discuss and analyze convergence analysis of DFV methods applied to a semilinear elliptic optimal control problems, for the sake of clarity in the presentation, highlighting the applicability and related results of the proposed DFV methods (also used in other Chapters) and in view of application of linear optimal elliptic problems; we first consider linear elliptic problems and carry out the convergence analysis. After establishment of convergence results for linear problems, we extend the proposed analysis to semilinear elliptic control problems.

Linear elliptic optimal control problems:

We discuss DFV approximations for distributed elliptic optimal control problems. The proposed control problems typically involve three unknown variables: control, state and costate. The state and costate are discretized by piecewise linear DFV methods, whereas, control discretization is based on three different approaches: variational discretization, piecewise constant and linear discretization. Since our discrete formulation is non-symmetric, we adopt *optimize-then-discretize* approach to approximate the control problem. Optimal *a priori* error estimates for all three variables in suitable norms are derived. Moreover, at the end of this Chapter, numerical experiments are conducted to support our theoretical findings.

2.1 Introduction (Linear)

A wide variety of processes in physical applications can be described by mathematical models which are based on elliptic constrained optimization. Probably, the most typical example is the stationary heating of a body by a controlled heat source. Problems of this kind arise if the body is heated by electromagnetic induction or by microwaves. Assuming the boundary temperature vanishes we can model the above process into the following optimization problem governed by elliptic equations with control u and state y.

$$\min_{u \in U_{ad}} J(y,u) := \frac{1}{2} \|y - y_d\|_{0,\Omega}^2 + \frac{\lambda}{2} \|u\|_{0,\Omega}^2 , \tag{2.1}$$

subject to

$$\left.\begin{aligned} -\nabla \cdot (\mathcal{A}\nabla y) = \mathcal{B}u + f, \quad \text{in} \quad \Omega, \\ y = 0 \quad \text{on} \quad \partial\Omega. \end{aligned}\right\} \tag{2.2}$$

The given data $f, y_d \in L^2(\Omega)$ or $H^1(\Omega)$, $\lambda > 0$ is a given regularization parameter, $\mathcal{B}$ is a continuous linear operator and $\mathcal{A} = (a_{ij}(x))_{2\times 2}$ denotes a real valued, symmetric and uniformly positive definite matrix in Ω, i.e., there exists a positive constant α_0 such that

$$\xi^T \mathcal{A}(x)\xi \geq \alpha_0 \xi^T \xi, \quad \forall \xi \in \mathbb{R}^2, \ \ \forall x \in \bar{\Omega}. \tag{2.3}$$

The set of admissible controls U_{ad} is defined by

$$U_{ad} = \{u \in L^2(\Omega) : u_a \leqslant u(x) \leqslant u_b, \, a.e. \, in \, \Omega\},$$

where the bounds $u_a, u_b \in \mathbb{R}$ and fulfill $u_a < u_b$. The control variable u represents a heat source which controls the temperature distribution via state equation (2.2). This heat source u is placed on the whole domain Ω. Due to limited heating and cooling capacities we can consider bounds on the control. The overall idea is to drive the state y as close as possible to the desired state y_d. The second term of the objective functional (2.1) penalizes excessive control cost.

For all $u \in U_{ad} \subset L^2(\Omega)$, there is exactly one associated state $y = y(u) \in H_0^1(\Omega) \cap H^2(\Omega)$ such that the mapping $\mathcal{G} : L^2(\Omega) \longrightarrow H_0^1(\Omega) \cap H^2(\Omega)$, $\mathcal{G} : u \mapsto y(u)$ is continuous. The solution operator is defined by $\mathcal{S} := E \circ \mathcal{G}$, where E denotes the continuous injection of $H^2(\Omega)$ in $L^2(\Omega)$. With the help of solution operator $\mathcal{S}$, we can transform the problem (2.1)-(2.2) into the control reduced quadratic optimization problem

$$\min_{u \in U_{ad}} j(u) := J(\mathcal{S}u, u) = \frac{1}{2} \|\mathcal{S}u - y_d\|_{0,\Omega}^2 + \frac{\lambda}{2} \|u\|_{0,\Omega}^2 . \tag{2.4}$$

Since the optimal control problem (2.4) is strictly convex, we can obtain the existence of a unique optimal solution. For the subsequent standard existence, uniqueness and first-order optimality results, we refer to [79]. In what follows, we denote the unique optimal control by u and the associated optimal state by $y = y(u)$. The first order necessary and sufficient optimality condition for (2.4) is given by the variational inequality

$$j'(u)(\tilde{u} - u) \geq 0, \quad \forall \tilde{u} \in U_{ad}, \tag{2.5}$$

where $j'(u) = \lambda u + \mathcal{S}^*(\mathcal{S}u - y_d)$ with $\mathcal{S}^*$ being the adjoint operator of $\mathcal{S}$. If we introduce an auxiliary function p defined by $\mathcal{B}^* p := \mathcal{S}^*(\mathcal{S}u - y_d)$ where, $\mathcal{B}^*$ is the adjoint operator of $\mathcal{B}$, then the optimality condition (2.5) can be re-formulated as

$$(\lambda u + \mathcal{B}^* p, \tilde{u} - u) \geq 0 \quad \forall \tilde{u} \in U_{ad}. \tag{2.6}$$

The function $p = p(u)$ is called *adjoint state* (or *costate*) associated with u and solves the *adjoint equation*

$$\left.\begin{aligned} -\nabla \cdot (\mathcal{A} \nabla p) = y - y_d \quad \text{in} \quad \Omega, \\ p = 0 \quad \text{on} \quad \partial\Omega. \end{aligned}\right\} \tag{2.7}$$

Introducing a pointwise projection on the admissible set U_{ad} as

$$P_{[u_a,u_b]} : L^2(\Omega) \longrightarrow U_{ad}, \quad P_{[u_a,u_b]}(z(x)) := max(u_a, min(u_b, z(x))),$$

the first order optimality condition (2.6) can be expressed as (see [42])

$$u(x) = P_{[u_a,u_b]}\left(\frac{-1}{\lambda}\mathcal{B}^* p(x)\right).$$

It follows directly from the definition above that the projection operator $P_{[u_a,u_b]}$ satisfies the following regularity properties

$$\left\|\nabla(P_{[u_a,u_b]}(z))\right\|_{L^\infty(\Omega)} \le \|\nabla z\|_{L^\infty(\Omega)}, \quad \forall z \in W^{1,\infty}(\Omega). \tag{2.8}$$

As far as numerical approximations of linear elliptic optimal control problems are concerned, the FE methods have been widely used. There are several results available in literature dealing with error analysis of FE approximation of linear elliptic optimal control problems; for instance, see [17, 36, 39, 41, 77]. Most of these publications analyzed the convergence of control in the L^2-norm and established the convergence order $\mathcal{O}(h)$ for piecewise constant discretizations. Whereas, with piecewise linear discretization of control, an order $\mathcal{O}(h^{3/2})$ was obtained. Recently, Hinze in [42] proposed a new approach called variational discretization, in which control is discretized implicitly by the projection of the discrete costate variable on the set of admissible controls and established the improved convergence order $\mathcal{O}(h^2)$ for control error. Keeping in mind desirable features of DG methods and FVE methods (mentioned in Chapter 1), we consider DFV approximations for state and costate equations, and in the light of [58, 59, 60], *optimize-then-discretize* approach is employed for the solvability of the linear optimal control problem.

We have arranged the remainder of this part in the following manner. Section 2.1 is introductory in nature. In Section 2.2 the DFV formulation of the proposed control problem is formulated. Therein, we present three different control discretization techniques: variational discretization, piecewise constant and linear discretization. In Section 2.3, we derive *a priori* error estimates for control, state and costate variables in suitable norms. Finally, in Section 2.4, we conduct some numerical experiments to substantiate the theoretical results of this part.

2.2 Discretization

In this Section, we present discontinuous finite volume schemes for the discretization of the optimal control problem. We also describe three different discretization techniques for control variable: variational discretization, piecewise linear and constant discretization.

2.2.1 Discontinuous finite volume discretizations

Let $\mathcal{T}_h$ denote a regular, quasi-uniform triangulation of $\bar{\Omega}$ into closed triangles K. Here h is the discretization parameter defined by setting $h = \max_{K \in \mathcal{T}_h} h_K$, where h_K is the diameter of the triangle K. Moreover, let $\mathcal{E}_h$ and $\mathcal{E}_h^{\Gamma}$ denote the set of all interior and boundary edges in $\mathcal{T}_h$, respectively. The dual partition $\mathcal{T}_h^*$ of the primal partition $\mathcal{T}_h$ is constructed in the following way. Each triangle $K \in \mathcal{T}_h$ is divided into three sub-triangles $(K_i^*)_{i=1}^3$ by joining the barycenter B of the triangle K to its vertices as shown in the Figure 2.1. In general, let K^* denote the dual element/control volume in $\mathcal{T}_h^*$. The union of these sub-triangles generated by the barycentric subdivison form the dual partition $\mathcal{T}_h^*$ of $\bar{\Omega}$.

We introduce the standard definitions of jumps and averages for scalar and vector functions as follows. Let e be an interior edge shared by two elements K_1 and K_2 in $\mathcal{T}_h$, and let $\mathbf{n_1}$ and $\mathbf{n_2}$ denote unit normal vectors on e pointing outward to K_1 and K_2, respectively. Then the average $\langle \cdot \rangle$ and jump $[\![\cdot]\!]$ on e for generic scalar q and vector $\mathbf{r}$ are defined respectively by

$$\langle q \rangle = \frac{1}{2}(q_1 + q_2), \ [\![q]\!] = q_1\mathbf{n_1} + q_2\mathbf{n_2} \text{ and } \langle \mathbf{r} \rangle = \frac{1}{2}(\mathbf{r_1} + \mathbf{r_2}), \ [\![\mathbf{r}]\!] = \mathbf{r_1}\cdot\mathbf{n_1} + \mathbf{r_2}\cdot\mathbf{n_2}.$$

Here, $q_i = (q|_{\partial K_i})$, $\mathbf{r}_i = (\mathbf{r}|_{\partial K_i})$. For $e \in \mathcal{E}_h^{\Gamma}$ with outward normal $\mathbf{n}$ we take $\langle q \rangle = q, \ [\![q]\!] = q\mathbf{n}, \ \langle \mathbf{r} \rangle = \mathbf{r}$ and $[\![\mathbf{r}]\!] = \mathbf{r}\cdot\mathbf{n}$.

The finite dimensional trial and test spaces associated with $\mathcal{T}_h$ and $\mathcal{T}_h^*$ are defined respectively, by

$$V_h = \{v_h \in L^2(\Omega) : v_h|_K \in \mathcal{P}_1(K) \quad \forall K \in \mathcal{T}_h\},$$
$$V_h^* = \{v_h \in L^2(\Omega) : v_h|_{K^*} \in \mathcal{P}_0(K^*) \quad \forall K^* \in \mathcal{T}_h^*\},$$

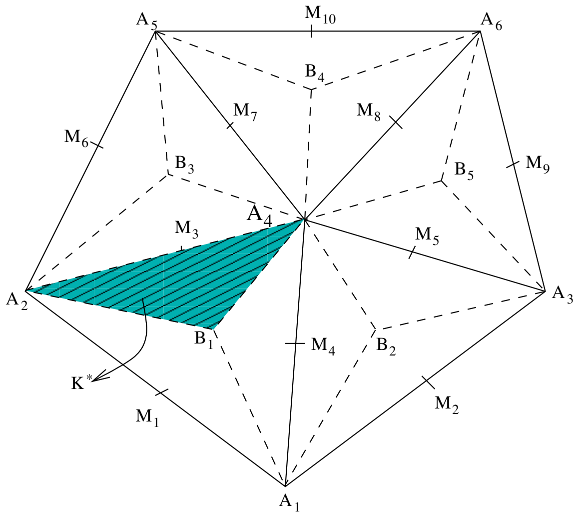


Figure 2.1: The dual partition of a triangulation.

where $\mathcal{P}_r(K)$ or $\mathcal{P}_r(K^*)$ denote the space of polynomials of degree less than or equal to r defined on the element K or K^*, respectively.

Let $V(h) = V_h + H^2(\Omega) \cap H_0^1(\Omega)$. Then the connection between trial and test spaces is characterized by the transfer operator $\gamma : V(h) \longrightarrow V_h^*$ which is defined by

$$\gamma v|_{K^*} = \frac{1}{h_e} \int_e v|_{K^*} ds, \quad \forall K^* \in \mathcal{T}_h^*,$$

where h_e represents the length of the edge e.

Some standard useful results satisfied by the map γ are collected in the following Lemma (for a proof, see [9, 49, 81]). These results will be used in the analysis throughout this thesis.

Lemma 2.2.1. *Let γ be the transfer operator. Then*

1. *γ satisfies the self-adjoint property with respect to the L^2-inner product, i.e.*

$$(v_h, \gamma q_h)_{0,\Omega} = (q_h, \gamma v_h)_{0,\Omega}, \quad \forall v_h, q_h \in V_h. \tag{2.9}$$

2. *For $v_h \in V_h$ if $|||v_h|||_0^2 := (v_h, \gamma v_h)$, then the norms $|||\cdot|||_0$ and $\|\cdot\|_{0,\Omega}$ are equivalent.*

3. *γ is stable with respect to norm $\|\cdot\|_{0,\Omega}$, i.e.*

$$\|\gamma v_h\|_{0,\Omega} = \|v_h\|_{0,\Omega}, \quad \forall v_h \in V_h. \tag{2.10}$$

4. *For all* $v \in V(h)$ *and* $K \in \mathcal{T}_h$*, we have*

$$\|v - \gamma v\|_{0,K} \leq Ch_K \|v\|_{1,K} \,. \tag{2.11}$$

Multiplying (2.2) by $\gamma v_h \in V_h^*$, integrating over the control volumes $K^* \in \mathcal{T}_h^*$, applying Gauss divergence Theorem and summing up over all control volumes, we obtain

$$-\sum_{K^*\in\mathcal{T}_h^*}\int_{\partial K^*} \mathcal{A}\nabla y \cdot \mathbf{n}\gamma v_h \, \mathrm{d}s = (\mathcal{B}u + f, \gamma v_h), \quad \forall v_h \in V_h,$$

where $\mathbf{n}$ denotes the unit normal vector to the boundary ∂K^* of K^*. Let $K_j^* \in \mathcal{T}_h^*(j = 1, 2, 3)$ be the three sub-triangles of triangle $K \in \mathcal{T}_h$, (see Figure 2.2). Then

$$\begin{aligned}\sum_{K^*\in\mathcal{T}_h^*}\int_{\partial K^*} \mathcal{A}\nabla y \cdot \mathbf{n}\gamma v_h \, \mathrm{d}s &= \sum_{j=1}^{3}\int_{\partial K_j^*} \mathcal{A}\nabla y \cdot \mathbf{n}\gamma v_h \, \mathrm{d}s \\ &= \sum_{j=1}^{3}\int_{A_{j+1}BA_j} \mathcal{A}\nabla y \cdot \mathbf{n}\gamma v_h \, \mathrm{d}s + \sum_{K\in\mathcal{T}_h}\int_{\partial K} \mathcal{A}\nabla y \cdot \mathbf{n}\gamma v_h \, \mathrm{d}s,\end{aligned} \tag{2.12}$$

where $A_4 = A_1$, see Figure 2.2.

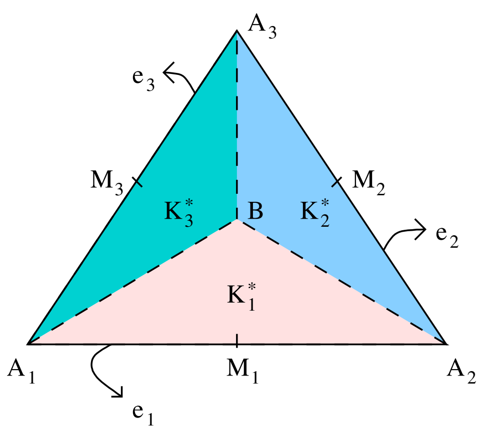


Figure 2.2: A triangular partition and its dual.

For any real numbers a, b, c and d, we have

$$ac - bd = \frac{1}{2}(a + b)(c - d) + \frac{1}{2}(a - b)(c + d). \tag{2.13}$$

Using (2.13) and the fact that $[\![\mathcal{A}\nabla y]\!] = 0$ in (2.12), we can obtain

$$\sum_{K^*\in\mathcal{T}_h^*}\int_{\partial K^*} \mathcal{A}\nabla y\cdot\mathbf{n}\gamma v_h\,\mathrm{d}s = \sum_{j=1}^{3}\int_{A_{j+1}BA_j} \mathcal{A}\nabla y\cdot\mathbf{n}\gamma v_h\,\mathrm{d}s + \sum_{e\in\mathcal{E}_h}\int_e [\![\gamma v_h]\!]\cdot\langle\mathcal{A}\nabla y\rangle\,\mathrm{d}s.$$

The discontinuous finite volume scheme corresponding to the state equation (2.2) is defined as: For a given u, find $y_h(u)\in V_h$ such that

$$A_h(y_h(u),v_h) + (\varphi(y_h(u)),\gamma v_h) = (\mathcal{B}u+f,\gamma v_h),\quad \forall v_h\in V_h, \tag{2.14}$$

where, the bilinear form $A_h(\cdot,\cdot): V_h\times V_h \longrightarrow \mathbb{R}$ is defined as (see [49])

$$\begin{aligned}A_h(v_h,q_h) = &-\sum_{K\in\mathcal{T}_h}\sum_{j=1}^{3}\int_{A_{j+1}BA_j} \mathcal{A}\nabla v_h\cdot\mathbf{n}\gamma q_h ds + \theta\sum_{e\in\mathcal{E}_h}\int_e\langle\mathcal{A}\nabla q_h\rangle\cdot[\![\gamma v_h]\!]\,\mathrm{d}s\\ &-\sum_{e\in\mathcal{E}_h}\int_e\langle\mathcal{A}\nabla v_h\rangle\cdot[\![\gamma q_h]\!]\,\mathrm{d}s + \sum_{e\in\mathcal{E}_h}\int_e\frac{\alpha}{h_e^{\beta}}[\![v_h]\!]\cdot[\![q_h]\!]\,\mathrm{d}s,\end{aligned} \tag{2.15}$$

where, α and β are penalty parameters independent of mesh size h. In general, $\theta\in[-1,1]$ and the cases $\theta=-1$, $\theta=0$, $\theta=1$, respectively correspond to the SIPG, IIPG, NIPG methods in the context of DG methods. As $[\![y]\!]=0$ and $[\![\gamma y]\!]=0$, the DFV scheme is consistent, i.e. the solution y satisfies

$$A_h(y,v_h) = (\mathcal{B}u+f,\gamma v_h)\quad \forall v_h\in V_h.$$

Now, we introduce the following natural mesh-dependent norm on space $V(h)$ which is naturally associated with the bilinear form $A_h(\cdot,\cdot)$:

$$|\!|\!| v_h |\!|\!|_h^2 := \sum_{K\in\mathcal{T}_h}|v_h|_{1,K}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-\beta}\,\|[\![v_h]\!]\|_{0,e}^2\,. \tag{2.16}$$

We have the following discrete Poincaré-Friedrichs type inequality (see [81])

$$\|v_h\| \le C\,|\!|\!| v_h |\!|\!|_h\,,\qquad \forall v_h\in V_h. \tag{2.17}$$

In addition, we have the following results which will be used in this thesis.

Lemma 2.2.2. *The bilinear form $A_h(\cdot,\cdot)$ defined in (2.15) possess the following well-known properties:*

1. *$A_h(\cdot,\cdot)$ is bounded and coercive with respect to $|||\cdot|||_h$, i.e., $\exists\, C > 0$ such that*

$$
\begin{aligned}
|A_h(v_h, q_h)| &\leq C\,|||v_h|||_h\,|||q_h|||_h\,, \quad &&\forall v_h, q_h \in V_h,\\
A_h(v_h, v_h) &\geq C\,|||v_h|||_h^2\,, &&\forall v_h \in V_h.
\end{aligned}
$$

   The proof can be obtained with the help of Lemma 2.2.1, for details kindly see [49].

2. *For all $v_h, q_h \in V_h$, the following relation holds*

$$
|A_h(v_h, q_h) - A_h(q_h, v_h)| \leq Ch\,|||v_h|||_h\,|||q_h|||_h\,. \tag{2.18}
$$

   For proof details cf. [81].

3. *Let $\epsilon_a(v_h, q_h) := a_h(v_h, q_h) - A_h(v_h, q_h)$. Then we have the following estimate*

$$
|\epsilon_a(v_h, q_h)| \leq Ch\,|||v_h|||_h\,|||q_h|||_h\,, \quad \forall\, v_h, q_h \in V_h, \tag{2.19}
$$

   where, the bilinear form $a_h(\cdot,\cdot)$ is defined by

$$
\begin{aligned}
a_h(v_h, q_h) = &\sum_{K\in\mathcal{T}_h}\int_K \mathcal{A}\nabla v_h\cdot\nabla q_h\,dx + \theta\sum_{e\in\mathcal{E}_h}\int_e [\![v_h]\!]\cdot\langle\mathcal{A}\nabla q_h\rangle ds\\
&+ \theta\sum_{e\in\mathcal{E}_h}\int_e [\![q_h]\!]\cdot\langle\mathcal{A}\nabla v_h\rangle ds + \sum_{e\in\mathcal{E}_h}\int_e \frac{\alpha}{h_e^{\beta}}[\![v_h]\!]\cdot[\![q_h]\!] ds.
\end{aligned}
$$

   For a proof we refer to Lemma 3.2 of [10].

2.2.2 Discretization of control

Choosing U_h as a a finite dimensional subspace of $L^2(\Omega)$, we introduce the discrete admissible set $U_{h,ad} = U_h \cap U_{ad}$. Now, we describe the following three approaches for the discretization of control variable

1. **Variational discretization.** (cf. [42]) The discretization in this case does not involve explicit control discretization. In this case, $U_h = L^2(\Omega)$ and therefore, the discrete admissible space $U_{h,ad}$ coincides with U_{ad}.

2. **Piecewise linear discretization.** Here, for the discretization of control variable, we choose the same space as for the discretization of the state variable i.e.,

$$U_h = \{u_h \in L^2(\Omega) : u_h|_K \in \mathcal{P}_1(K) \quad \forall K \in \mathcal{T}_h\}.$$

3. **Piecewise constant discretization**. Another possibility is to use piecewise constant functions for the discretization of control variable. In this case, the discrete control space is defined by

$$U_h = \{u_h \in L^2(\Omega) : u_h|_K \in P_0(K) \quad \forall K \in \mathcal{T}_h\}.$$

On applying the DFV scheme to discretize the state and costate equations directly, the discrete formulation of the elliptic optimal control problem (2.1)-(2.2) is given by : Find $(y_h, p_h, u_h) \in V_h \times V_h \times U_{h,ad}$ such that

$$A_h(y_h, v_h) = (\mathcal{B}u_h + f, \gamma v_h), \quad \forall v_h \in V_h, \tag{2.20}$$

$$A_h(p_h, q_h) = (y_h - y_d, \gamma q_h), \quad \forall q_h \in V_h, \tag{2.21}$$

$$(\lambda u_h + \mathcal{B}^* p_h, \tilde{u}_h - u_h) \geq 0, \quad \forall \tilde{u}_h \in U_{h,ad}. \tag{2.22}$$

2.3 Error estimates

In this Section, we derive *a priori* error estimates for state, costate and control variables with different control discretization approaches: variational discretization, piecewise constant and linear discretization. To begin with, we consider the following auxiliary equations. For any arbitrary u, let $y_h(u)$ be the solution of

$$A_h(y_h(u), v_h) = (\mathcal{B}u + f, \gamma v_h), \quad \forall v_h \in V_h, \tag{2.23}$$

and for any arbitrary $y = y(u)$, let $p_h(y)$ be the solution of

$$A_h(p_h(y), q_h) = (y - y_d, \gamma q_h), \quad \forall q_h \in V_h. \tag{2.24}$$

Noting that $y_h = y_h(u_h)$ and $p_h = p_h(y_h)$, we have the following intermediate estimates which will be used in our forthcoming analysis.

Lemma 2.3.1. *Let $y_h(u)$ and $p_h(y)$ be the solutions of (2.23) and (2.24), respectively. Then there exists a positive constant C independent of h such that*

$$|||y_h(u) - y_h|||_h \leq C \, \|u - u_h\|_{0,\Omega} \, , \quad |||p_h(y) - p_h|||_h \leq C \, \|y - y_h\|_{0,\Omega} \, .$$

Proof. The subtraction of equation (2.20) from equation (2.23) implies the relation

$$A_h(y_h(u) - y_h, v_h) = (\mathcal{B}(u - u_h), \gamma v_h), \quad \forall v_h \in V_h.$$

On choosing $v_h = y_h(u) - y_h$ in the above relation and utilizing the ellipticity of the bilinear form $A_h(\cdot,\cdot)$ and the continuity of operator $\mathcal{B}$, we arrive at

$$|||y_h(u) - y_h|||_h^2 \leq C \, \|u - u_h\|_{0,\Omega} \, \|\gamma(y_h(u) - y_h)\|_{0,\Omega} \, ,$$

which on application of results (2.10) and (2.17) gives the estimate

$$|||y_h(u) - y_h|||_h \leq C \, \|u - u_h\|_{0,\Omega} \, .$$

The second required estimate can be readily obtained by subtracting (2.21) from (2.24), choosing $q_h = p_h(y) - p_h$ and proceeding analogously as above. □

The approximation properties of the DFV solution of second order elliptic problems are collected in the following Theorem. For a detailed proof, we refer to Theorem 3.1 and Theorem 3.2 established in [49].

Lemma 2.3.2. *Let $y_h(u)$ and $p_h(y)$ be the solutions of (2.23) and (2.24), respectively. Then there exists a constant C independent of h such that*

$$|||y - y_h(u)|||_h + |||p - p_h(y)|||_h \leq Ch, \tag{2.25}$$

Further, if we assume that $\mathcal{A} \in W^{2,\infty}(\Omega)$ and $u, f, y_d \in H^1(\Omega)$, then

$$\|y - y_h(u)\|_{0,\Omega} + \|p - p_h(y)\|_{0,\Omega} \leq Ch^2. \tag{2.26}$$

To derive the estimates for the control, state and costate errors in L^2-norm, we will make use of the following Lemma.

Lemma 2.3.3. *Let $\mathcal{A} \in W^{2,\infty}(\Omega)$ and $u, f, y_d \in H^1(\Omega)$. Then we have*

$$(p - p_h, \mathcal{B}(u_h - u)) \leq Ch^2 \|u - u_h\|_{0,\Omega} + Ch \|u - u_h\|_{0,\Omega}^2 .$$

Proof. We can write

$$\begin{aligned}(p - p_h, \mathcal{B}(u_h - u)) =& [(p - p_h(y), \mathcal{B}(u_h - u)) + (p_h(y) - p_h - \gamma(p_h(y) - p_h), \\ & \mathcal{B}(u_h - u))] + (\gamma(p_h(y) - p_h), \mathcal{B}(u_h - u)) = I_1 + I_2. \quad (2.27)\end{aligned}$$

Applying Cauchy-Schwarz inequality, using the result (2.26) and the approximation property of transfer operator γ, we have

$$\begin{aligned} I_1 &\leq C \|p - p_h(y)\|_{0,\Omega} \|u_h - u\|_{0,\Omega} + \|p_h(y) - p_h - \gamma(p_h(y) - p_h)\|_{0,\Omega} \|u_h - u\|_{0,\Omega} \\ &\leq Ch^2 \|u_h - u\|_{0,\Omega} + Ch \| p_h(y) - p_h \|_h \|u_h - u\|_{0,\Omega} \end{aligned}$$

Utilizing the estimates of Lemma 2.3.1 in the above relation implies

$$\begin{aligned} I_1 &\leq Ch^2 \|u_h - u\|_{0,\Omega} + Ch \|y - y_h\|_{0,\Omega} \|u_h - u\|_{0,\Omega} \\ &\leq Ch^2 \|u_h - u\|_{0,\Omega} + Ch \left(\|y - y_h(u)\|_{0,\Omega} + \|y_h(u) - y_h\|_{0,\Omega} \right) \|u_h - u\|_{0,\Omega} \\ &\leq Ch^2 \|u_h - u\|_{0,\Omega} + Ch \|u_h - u\|_{0,\Omega}^2 , \end{aligned}$$

where the last inequality follows from the application of property (2.17) and result (2.26). Using the state equations (2.20) and (2.23), we can express the second term of (2.27) as

$$\begin{aligned} I_2 =& [A_h(y_h - y_h(u), p_h(y) - p_h) - A_h(p_h(y) - p_h, y_h - y_h(u))] \\ & + (y - y_h(u), \gamma(y_h - y_h(u))) + (y_h(u) - y_h, \gamma(y_h - y_h(u))) \\ \leq& Ch \| y_h - y_h(u) \|_h \| p_h(y) - p_h \|_h + Ch^2 \|y_h - y_h(u)\|_{0,\Omega} , \end{aligned}$$

where the last inequality follows from the estimate (2.18), the result (2.26), the stability (2.10) of γ in $\|\cdot\|_{0,\Omega}$ and using the fact that $(y_h - y_h(u), \gamma(y_h - y_h(u))) \geq 0$. The property (2.17) alongwith the estimate of Lemma 2.3.1 and result (2.26) in the above inequality gives the relation

$$I_2 \leq Ch^2 \|u_h - u\|_{0,\Omega} + Ch \|u_h - u\|_{0,\Omega}^2 .$$

Plugging the bounds of the terms I_1 and I_2 in (2.27), we can obtain

$$(p - p_h, \mathcal{B}(u_h - u)) \leq Ch^2 \|u - u_h\|_{0,\Omega} + Ch \|u - u_h\|_{0,\Omega}^2 .$$

□

2.3.1 Error estimates for control

In this Section, we will derive the estimates for $\|u - u_h\|_{0,\Omega}$ with the three different discretization approaches for control described in Section 2.2. We start with variational discretization approach.

With variational discretization:

Here, we derive the convergence result for the control error in L^2-norm in the case of no explicit control discretization; see Section 2.2.2. In this case, $U_h = L^2(\Omega)$, and therefore $U_{h,ad} = U_{ad}$.

Theorem 2.3.4. *Assume that $\mathcal{A} \in W^{2,\infty}(\Omega)$ and $u, f, y_d \in H^1(\Omega)$. Let $(y, p, u) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega)) \times U_{ad}$ be the exact solutions and $(y_h, p_h, u_h) \in V_h \times V_h \times U_{ad}$ be the solutions of (2.20)-(2.22) with variational discretization approach. Then there exists a constant $C > 0$ independent of mesh size h such that*

$$\|u - u_h\|_{0,\Omega} \leq Ch^2.$$

Proof. The discrete variational inequality for this approach is

$$(\lambda u_h + \mathcal{B}^* p_h, \tilde{u} - u_h) \geq 0, \quad \forall \tilde{u} \in U_{ad}. \tag{2.28}$$

An application of continuous (2.6) and discrete (2.28) variational inequalities yields

$$(\lambda u + \mathcal{B}^* p, u_h - u) + (\lambda u_h + \mathcal{B}^* p_h, u - u_h) \geq 0.$$

This implies that

$$\lambda \left\| u - u_h \right\|_{0,\Omega}^2 \leq (\mathcal{B}^*(p - p_h), u_h - u) = (p - p_h, \mathcal{B}(u_h - u)).$$

The estimate of Lemma 2.3.3 in the above relation yields the result

$$\left\| u - u_h \right\|_{0,\Omega} \leq Ch^2. \tag{2.29}$$

This completes the proof. □

With piecewise constant discretization:

In this Section, we are going to derive an estimate for the error $\left\| u - u_h \right\|_{0,\Omega}$ when the control is discretized by element-wise constant polynomials; see Section 2.2.2. To formulate the main result of this Section we will utilize the idea presented in [17] for elliptic optimal control problem. We introduce the L^2-projection operator $\Pi_0 : L^2(\Omega) \longrightarrow U_h$ with the following approximation property

$$\left\| v - \Pi_0 v \right\|_{0,K} \leq Ch \left\| v \right\|_{1,K}. \tag{2.30}$$

The error estimate for control reads as follows.

Theorem 2.3.5. *Let* (y, p, u) *be the exact solution of problem* (2.1)-(2.2) *and* (y_h, p_h, u_h) *be the solution of discrete problem* (2.20)-(2.22) *under piecewise constant control discretization. Then we can obtain the following result*

$$\left\| u - u_h \right\|_{0,\Omega} \leq Ch.$$

Proof. We note that due to piecewise constant discretization, we have the property that $\Pi_0 U_{ad} \subset U_{h,ad}$. Therefore, the continuous and discrete optimality conditions imply

$$(\lambda u + \mathcal{B}^* p, u_h - u) + (\lambda u_h + \mathcal{B}^* p_h, \Pi_0 u - u_h) \geq 0.$$

Adding and subtracting u in the second term and rearranging we can obtain

$$\lambda \|u - u_h\|_{0,\Omega}^2 \leq (\mathcal{B}^*(p - p_h), u_h - u) + (\lambda u_h + \mathcal{B}^* p_h, \Pi_0 u - u).$$

Since Π_0 is an orthogonal projection and $u_h \in U_{h,ad}$, the term $(\lambda u_h, \Pi_0 u - u)$ in the above relation vanishes. Therefore, we have

$$\lambda \|u - u_h\|_{0,\Omega}^2 \leq \underbrace{(\mathcal{B}^*(p - p_h), u_h - u)}_{(\star)} + \underbrace{(\mathcal{B}^* p_h, \Pi_0 u - u)}_{(\star\star)}. \tag{2.31}$$

The first term of (2.31) can be bounded by directly applying the estimates of Lemma 2.3.3, i.e.,

$$(\star) \leq Ch^2 \|u - u_h\|_{0,\Omega} + Ch \|u - u_h\|_{0,\Omega}^2 \leq Ch^2 + Ch \|u - u_h\|_{0,\Omega}^2.$$

Whereas for the second term of (2.31), we use the orthogonality and approximation property (2.30) of Π_0 to obtain

$$\begin{aligned}(\star\star) = (\mathcal{B}^* p_h - \Pi_0 \mathcal{B}^* p_h, \Pi_0 u - u) &\leq \|\mathcal{B}^* p_h - \Pi_0 \mathcal{B}^* p_h\|_{0,\Omega} \|\Pi_0 u - u\|_{0,\Omega} \\ &\leq Ch \|p_h\|_h \|\Pi_0 u - u\|_{0,\Omega}.\end{aligned}$$

Now, it is left to show that p_h is uniformly bounded. From the discrete costate equation (2.21) and the coercivity of $A_h(\cdot,\cdot)$, we can obtain

$$\|p_h\|_h \leq C\left(\|y_h\|_{0,\Omega} + \|y_d\|_{0,\Omega}\right) \leq C\left(\|y_h\|_h + \|y_d\|_{0,\Omega}\right) \tag{2.32}$$

Similarly, from the discrete state equation (2.20) and the coercivity of $A_h(\cdot,\cdot)$, we have the relation $\|y_h\|_h \leq C\left(\|u_h\|_{0,\Omega} + \|f\|_{0,\Omega}\right)$. Using this relation and the uniform boundedness of $U_{h,ad}$ in (2.32) implies that p_h is uniformly bounded. Finally, substituting the bounds of the terms in (2.31), and using the estimate (2.30) the desired convergence result follows. □

With piecewise linear discretization:

In this Section, we derive the error $\|u - u_h\|_{0,\Omega}$ with the discretization of control by the piecewise linear functions as described in Section 2.2.2. The error analysis is based on an assumption on the structure of the active sets. We start by grouping each element $K \in \mathcal{T}_h$ into three sets $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2 \cup \mathcal{T}_h^3$ with $\mathcal{T}_h^i \cap \mathcal{T}_h^j = \emptyset$ for $i \neq j$ according to the value of $u(x)$ on K and define these disjoint sets by

$$\left.\begin{aligned} \mathcal{T}_h^1 &= \{K \in \mathcal{T}_h : u(x) = u_a \quad or \quad u(x) = u_b, \quad \forall x \in K\}, \\ \mathcal{T}_h^2 &= \{K \in \mathcal{T}_h : u_a < u(x) < u_b \quad \forall x \in K\}, \\ \mathcal{T}_h^3 &= \mathcal{T}_h \setminus (\mathcal{T}_h^1 \cup \mathcal{T}_h^2). \end{aligned}\right\} \tag{2.33}$$

The following assumption is made which will be utilized in our subsequent analysis.

Assumption 2.3.6. There exists a constant $C > 0$ independent of the mesh size h such that

$$\sum_{K \in \mathcal{T}_h^3} |K| \leq Ch.$$

A function $\tilde{u}_h(x) \in U_{h,ad}$ on an arbitrary triangle $K \in \mathcal{T}_h$ is defined by

$$\tilde{u}_h(x) = \begin{cases} u_a & \text{if} \min_{x \in K} u(x) = u_a, \\ u_b & \text{if} \max_{x \in K} u(x) = u_b, \\ \tilde{I}_h u & \text{else}, \end{cases} \tag{2.34}$$

where $\tilde{I}_h u$ is the Lagrange interpolate of u. To avoid any ambiguity, h is chosen small enough such that $\min_{x \in K} u(x) = u_a$ and $\max_{x \in K} u(x) = u_b$ cannot happen simultaneously in the same K. It follows directly from the above definition that for any $\tilde{u}_h \in U_{h,ad}$ we have (cf. [17, Lemma 2.1])

Lemma 2.3.7.

$$(\lambda u + \mathcal{B}^* p, \tilde{u} - \tilde{u}_h) \geq 0, \quad \forall \tilde{u} \in U_{ad}. \tag{2.35}$$

For our subsequent analysis, we will exploit the following assertion (a proof can be found in [62]).

Lemma 2.3.8. *Let u be the solution of the continuous problem (2.1)-(2.2). Then, if the assumption (2.3.6) is fulfilled and $p \in W^{1,\infty}(\Omega)$, the following assertion*

$$|(\lambda u + \mathcal{B}^* p, \tilde{u}_h - u)| \leq \frac{C}{\lambda} h^3 \|\nabla p\|^2_{L^\infty(\Omega)}, \quad \forall \tilde{u}_h \in U_{h,ad},$$

holds true for a positive constant C independent of h.

The main result in this Section is stated as follows.

Theorem 2.3.9. *Let (y, p, u) be the solution of (2.1)-(2.2) and (y_h, p_h, u_h) be the solution of discrete problem (2.20)-(2.22) under the piecewise linear discretization for the control variable. Then the following result holds:*

$$\|u - u_h\|_{0,\Omega} \leq Ch^{3/2}.$$

Proof. The continuous (2.6) and discrete variational (2.22) inequalities under the piecewise linear control discretization leads to the relation

$$(\lambda u + \mathcal{B}^* p, u_h - u) + (\lambda u_h + \mathcal{B}^* p_h, \tilde{u}_h - u_h) \geq 0.$$

Splitting $u_h - u = (\tilde{u}_h - u) + (u_h - \tilde{u}_h)$ in the first term of the above relation yields

$$\lambda(u - u_h, u_h - \tilde{u}_h) + (\mathcal{B}^*(p - p_h), u_h - \tilde{u}_h) + (\lambda u + \mathcal{B}^* p.\tilde{u}_h - u) \geq 0,$$

After rearranging the terms we can obtain the following inequality

$$\begin{aligned} \lambda \|u - u_h\|^2_{0,\Omega} \leq& \lambda(u - u_h + \mathcal{B}^*(p - p_h), u - \tilde{u}_h) + (p - p_h, \mathcal{B}(u_h - u)) \\ &+ (\lambda u + \mathcal{B}^* p, \tilde{u}_h - u). \end{aligned} \tag{2.36}$$

We now apply Cauchy-Schwarz inequality in the first term to readily have

$$\begin{aligned} \lambda \|u - u_h\|^2_{0,\Omega} \leq& \left(\lambda \|u - u_h\|_{0,\Omega} + \|p - p_h\|_{0,\Omega}\right) \|u - \tilde{u}_h\|_{0,\Omega} + (p - p_h, \mathcal{B}(u_h - u)) \\ &+ |(\lambda u + \mathcal{B}^* p, \tilde{u}_h - u)|. \end{aligned} \tag{2.37}$$

From triangle inequality and estimates (2.26) we can obtain $\|p - p_h\|_{0,\Omega} \leq Ch^2 +$

$C\|u-u_h\|_{0,\Omega}$. Therefore, the relation (2.37) can be rewritten as

$$\lambda \|u-u_h\|_{0,\Omega}^2 \leq \left(Ch^2 + C\|u-u_h\|_{0,\Omega}\right)\|u-\tilde{u}_h\|_{0,\Omega} + (p-p_h, \mathcal{B}(u_h-u)) \\ + |(\lambda u + \mathcal{B}^* p, \tilde{u}_h - u)|. \tag{2.38}$$

Now, to estimate the term $\|u-\tilde{u}_h\|_{0,\Omega}$, we express it as

$$\|u-\tilde{u}_h\|_{0,\Omega}^2 = \sum_{K\in\mathcal{T}_h} \|u-\tilde{u}_h\|_{L^2(K)}^2 = \sum_{K\in\mathcal{T}_h^2} \|u-\tilde{u}_h\|_{L^2(K)}^2 + \sum_{K\in\mathcal{T}_h^3} \|u-\tilde{u}_h\|_{L^2(K)}^2 \\ = T_1 + T_2.$$

where we have used the fact that $\tilde{u}_h = u$ on $\mathcal{T}_h^1$, and hence $\sum_{K\in\mathcal{T}_h^1} \|u-\tilde{u}_h\|_{L^2(K)}^2 = 0$. In order to bound T_1 we use the relation $u = \frac{-1}{\lambda}\mathcal{B}^* p$ on all triangles $K \in \mathcal{T}_h^2$, to obtain

$$\sum_{K\in\mathcal{T}_h^2} \left\|u-\tilde{I}_h u\right\|_{L^2(K)}^2 \leq Ch^4 \sum_{K\in\mathcal{T}_h^2} \|\nabla^2 u\|_{L^2(K)}^2 \leq \frac{C}{\lambda^2} h^4 \|\nabla^2 p\|_{0,\Omega}^2,$$

whereas to estimate the second term T_2, we employ the projection property (2.8) together with the assumption (2.3.6) to get

$$\sum_{K\in\mathcal{T}_h^3} \left\|u-\tilde{I}_h u\right\|_{L^2(K)}^2 \leq C \sum_{K\in\mathcal{T}_h^3} |K| \left\|u-\tilde{I}_h u\right\|_{L^\infty(K)}^2 \\ \leq Ch^3 \|\nabla u\|_{L^\infty(\Omega)}^2 \leq \frac{C}{\lambda^2} h^3 \|\nabla p\|_{L^\infty(\Omega)}^2.$$

Plugging the bounds of T_1 and T_2 in (2.39) we arrive at

$$\|u-\tilde{u}_h\|_{0,\Omega} = \mathcal{O}(h^{3/2}). \tag{2.39}$$

Using the result (2.39) and the estimates of Lemmas 2.3.3 and 2.3.8 in (2.36) and applying Young's inequality, we can readily obtain the required result

$$\|u-u_h\|_{0,\Omega} = \mathcal{O}(h^{3/2}).$$

□

2.3.2 Error estimates for state and costate

In this Section, we will derive the optimal convergence results for state and costate error in L^2-norm and mesh-dependent norm.

Under variational discretization of control:

Theorem 2.3.10. *Let u be the optimal control of (2.4) with associated state y and costate p, and let (u_h, y_h, p_h) be their DFV approximations with variational discretization approach. Then the following result holds:*

$$\|y - y_h\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \leq Ch^2.$$

Proof. For finding the estimates for state and costate error in L^2-norm, we split the state and costate errors as $y - y_h = (y - y_h(u)) + (y_h(u) - y_h)$ and $p - p_h = (p - p_h(y)) + (p_h(y) - p_h)$, respectively. Then from triangle inequality we have

$$\|y - y_h\|_{0,\Omega} \leq \|y - y_h(u)\|_{0,\Omega} + \|y_h(u) - y_h\|_{0,\Omega},$$
$$\|p - p_h\|_{0,\Omega} \leq \|p - p_h(y)\|_{0,\Omega} + \|p_h(y) - p_h\|_{0,\Omega}.$$

The property (2.17) and estimates of Lemma 2.3.1 implies that

$$\|y - y_h\|_{0,\Omega} \leq \|y - y_h(u)\|_{0,\Omega} + \|u - u_h\|_{0,\Omega}, \tag{2.40}$$
$$\|p - p_h\|_{0,\Omega} \leq \|p - p_h(y)\|_{0,\Omega} + \|y - y_h\|_{0,\Omega}. \tag{2.41}$$

Applying the results (2.26) and (2.29) in (2.40), we can easily obtain

$$\|y - y_h\|_{0,\Omega} \leq Ch^2. \tag{2.42}$$

Using the above estimate (2.42) and $\|p - p_h(y)\|_{0,\Omega} \leq Ch^2$ in the relation (2.41), we immediately get $\|p - p_h\|_{0,\Omega} \leq Ch^2$. This completes the proof. □

Under explicit discretization of control:

In this Section, we derive the estimates for errors $\|y - y_h\|_{0,\Omega}$ and $\|y - y_h\|_{0,\Omega}$, when the piecewise constant or linear discretization for the control variable is employed. We note that unlike the case of variational discretization approach, the derivation of optimal order L^2-error estimates for state and costate variables is more involved and sophisticated. To establish the $\mathcal{O}(h^2)$ convergence, we appeal to duality argument. The following Theorem provides the error estimates for state and costate variables under piecewise constant or linear control discretization.

Theorem 2.3.11. *Let u be the optimal control of (2.4) with associated state y and costate p, and let (u_h, y_h, p_h) be their DFV approximations under piecewise linear (or piecewise constant) discretization of the control variable. Then the following error estimates hold:*

$$\|y - y_h\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \leq Ch^2.$$

Proof. We start by splitting the total error and applying triangle inequality as:

$$\|y - y_h\|_{0,\Omega} \leq \|y - y_h(u)\|_{0,\Omega} + \|y_h(u) - y_h(\Pi_h u)\|_{0,\Omega} + \|y_h(\Pi_h u) - y_h\|_{0,\Omega}, \quad (2.43)$$

where Π_h represents the L^2-projection operator onto the discrete control space U_h. Next, let $\tilde{p}_h \in V_h$ be the unique solution of the auxiliary discrete dual elliptic problem

$$A_h(\tilde{p}_h, z_h) = (y_h(u) - y_h(\Pi_h u), \gamma z_h), \quad \forall z_h \in V_h, \quad (2.44)$$

Let us choose $z_h = \tilde{p}_h$ in (2.44). Then from the coercivity of $A_h(\cdot,\cdot)$ and property (2.17), we arrive at the following relation

$$|||\tilde{p}_h|||_h \leq C \, \|y_h(u) - y_h(\Pi_h u)\|_{0,\Omega}. \quad (2.45)$$

Now, on choosing $z_h = y_h(u) - y_h(\Pi_h u)$ in (2.44), we have

$$A_h(\tilde{p}_h, y_h(u) - y_h(\Pi_h u)) = (y_h(u) - y_h(\Pi_h u), \gamma(y_h(u) - y_h(\Pi_h u))). \quad (2.46)$$

Employing the discrete state equation for $y_h(u)$ and $y_h(\Pi_h u)$, we can obtain

$$A_h(y_h(u) - y_h(\Pi_h u), \tilde{p}_h) = (\mathcal{B}(u - \Pi_h u), \gamma\tilde{p}_h). \tag{2.47}$$

We then proceed to subtract (2.47) from (2.46) and using the definition of the norm $\|\!|\cdot|\!\|_{0,h}$ and its equivalence with the norm $\|\cdot\|_{0,\Omega}$, to arrive at

$$\begin{aligned}\|y_h(u) - y_h(\Pi_h u)\|_{0,\Omega}^2 \leq & |A_h(\tilde{p}_h, y_h(u) - y_h(\Pi_h u)) - A_h(y_h(u) - y_h(\Pi_h u), \tilde{p}_h)| \\ & + (\mathcal{B}(u - \Pi_h u), \gamma\tilde{p}_h).\end{aligned}$$

By virtue of the properties of projection operator Π_h applied in the above inequality, we can assert that

$$\begin{aligned}\|y_h(u) - y_h(\Pi_h u)\|_{0,\Omega}^2 \leq & (\mathcal{B}(u - \Pi_h u), \gamma\tilde{p}_h - \tilde{p}_h) + (u - \Pi_h u, \mathcal{B}^*\tilde{p}_h - \Pi_h\mathcal{B}^*\tilde{p}_h) \\ & + |A_h(\tilde{p}_h, y_h(u) - y_h(\Pi_h u)) - A_h(y_h(u) - y_h(\Pi_h u), \tilde{p}_h)| \\ = & S_1 + S_2 + S_3.\end{aligned} \tag{2.48}$$

Approximation properties of γ and the L^2-projection Π_h, and a direct application of (2.45) readily yield appropriate bounds for S_1 and S_2, respectively:

$$S_1 + S_2 \leq Ch \|u - \Pi_h u\|_{0,\Omega} \|\!|\tilde{p}_h|\!\|_h \leq Ch \|u - \Pi_h u\|_{0,\Omega} \|y_h(u) - y_h(\Pi_h u)\|_{0,\Omega}.$$

We next use the result (2.18) and relation (2.45) to obtain

$$S_3 \leq Ch \|\!|y_h(u) - y_h(\Pi_h u)|\!\|_h \|\!|\tilde{p}_h|\!\|_h \leq Ch \|u - \Pi_h u\|_{0,\Omega} \|y_h(u) - y_h(\Pi_h u)\|_{0,\Omega},$$

where we have used the relation $\|\!|y_h(u) - y_h(\Pi_h u)|\!\|_h \leq C \|u - \Pi_h u\|_{0,\Omega}$ which is an analogue of the results of Lemma 2.3.1. Finally substituting the bounds of S_1, S_2 and S_3 in (2.48), one straightforwardly arrives at

$$\|y_h(u) - y_h(\Pi_h u)\|_{0,\Omega} \leq Ch \|u - \Pi_h u\|_{0,\Omega}. \tag{2.49}$$

For the third term in (2.43) we proceed similarly as in the proof of Lemma 2.3.1 to obtain

$$\|y_h(\Pi_h u) - y_h\|_{0,\Omega} \leq \|\!|y_h(\Pi_h u) - y_h|\!\|_h \leq C \|\Pi_h u - u_h\|_{0,\Omega}. \tag{2.50}$$

Using the discrete variational inequality along with the projection property of Π_h and (2.35), we have the following relation

$$
\begin{aligned}
\lambda \left\|\Pi_h u - u_h\right\|_{0,\Omega}^2 &= \lambda(u - u_h, \Pi_h u - u_h) \\
&\le (\mathcal{B}^*(p - p_h), u_h - \Pi_h u) \\
&= ((p - p_h(u)), \mathcal{B}(u_h - \Pi_h u)) + (p_h(u) - p_h(y_h(\Pi_h u)), \mathcal{B}(u_h - \Pi_h u)) \\
&\quad + (p_h(y_h(\Pi_h u)) - p_h, \mathcal{B}(u_h - \Pi_h u)) \\
&= (p - p_h(u), \mathcal{B}(u_h - \Pi_h u)) + (p_h(u) - p_h(y_h(\Pi_h u)), \mathcal{B}(u_h - \Pi_h u)) \\
&\quad + (p_h(y_h(\Pi_h u)) - p_h - \gamma(p_h(y_h(\Pi_h u)) - p_h), \mathcal{B}(u_h - \Pi_h u)) \\
&\quad + (\gamma(p_h(y_h(\Pi_h u)) - p_h), \mathcal{B}(u_h - \Pi_h u)) \\
&= J_1 + J_2 + J_3 + J_4. \qquad (2.51)
\end{aligned}
$$

Using Cauchy-Schwarz inequality and (2.26) to bound the first term, we have

$$J_1 \le \left\|p - p_h(u)\right\|_{0,\Omega} \left\|u_h - \Pi_h u\right\|_{0,\Omega} \le Ch^2 \left\|u_h - \Pi_h u\right\|_{0,\Omega}.$$

For J_2 an application of Lemma 2.3.1 and (2.49) yields

$$J_2 \le \left\|y_h(u) - y_h(\Pi_h u)\right\|_{0,\Omega} \left\|u_h - \Pi_h u\right\|_{0,\Omega} \le Ch \left\|u - \Pi_h u\right\|_{0,\Omega} \left\|u_h - \Pi_h u\right\|_{0,\Omega}.$$

To bound the third term we use approximation property of γ and result of Lemma 2.3.1

$$
\begin{aligned}
J_3 &\le Ch \left|\!\left|\!\left|p_h(y_h(\Pi_h u)) - p_h\right|\!\right|\!\right|_h \left\|u_h - \Pi_h u\right\|_{0,\Omega} \\
&\le Ch \left\|y_h(\Pi_h u) - y_h\right\|_{0,\Omega} \left\|u_h - \Pi_h u\right\|_{0,\Omega} \le Ch \left\|u_h - \Pi_h u\right\|_{0,\Omega}^2.
\end{aligned}
$$

Proceeding analogously to the proof of Lemma 2.3.3 and using result (2.18), the last term of the expression (2.51) can be estimated as

$$
\begin{aligned}
J_4 &\le A_h(y_h - y_h(\Pi_h u), p_h(\Pi_h u) - p_h) - A_h(p_h(\Pi_h u) - p_h, y_h - y_h(\Pi_h u)) \\
&\le Ch \left|\!\left|\!\left|y_h - y_h(\Pi_h u)\right|\!\right|\!\right|_h \left|\!\left|\!\left|p_h(\Pi_h u) - p_h\right|\!\right|\!\right|_h \le Ch \left\|u_h - \Pi_h u\right\|_{0,\Omega}^2.
\end{aligned}
$$

Inserting the bounds for J_1, J_2, J_3 and J_4 in (2.51), plugging (2.49) and (2.50) into (2.43), using interpolation estimates, along with (2.26), we obtain an optimal estimate for the state error

$$\|y - y_h\|_{0,\Omega} = \mathcal{O}(h^2). \tag{2.52}$$

Finally, splitting the costate velocity error as $p - p_h = (p - p_h(y)) + (p_h(y) - p_h)$, using the triangle inequality, Lemma 2.3.1, relations (2.26) and (2.52), we get the second desired estimate

$$\begin{aligned}\|p - p_h\|_{0,\Omega} &\le \|p - p_h(y)\|_{0,\Omega} + \|p_h(y) - p_h\|_{0,\Omega} \\ &\le \|p - p_h(y)\|_{0,\Omega} + \|y - y_h\|_{0,\Omega} = \mathcal{O}(h^2).\end{aligned}$$

□

In mesh-dependent norm:

Theorem 2.3.12. *Let u be the optimal control of (2.1)-(2.2) with associated state y and costate p and (y_h, p_h, u_h) be the solutions of discrete problem (2.20)-(2.22). Then the following result holds:*

$$|||y - y_h|||_h + |||p - p_h|||_h \le Ch. \tag{2.53}$$

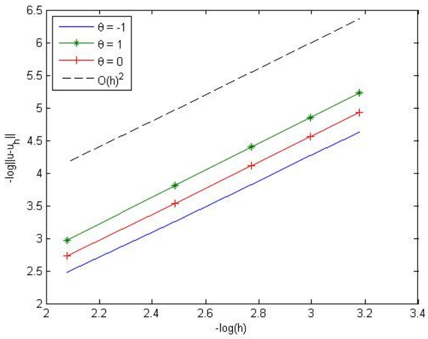
Proof. From the triangle inequality and Lemma 2.3.1, we arrive at

$$|||y - y_h|||_h \le |||y - y_h(u)|||_h + |||y_h(u) - y_h|||_h \le |||y - y_h(u)|||_h + C\,\|u - u_h\|_{0,\Omega}$$
$$|||p - p_h|||_h \le |||p - p_h(y)|||_h + |||p_h(y) - p_h|||_h \le |||p - p_h(y)|||_h + C\,\|y - y_h\|_{0,\Omega}\,.$$

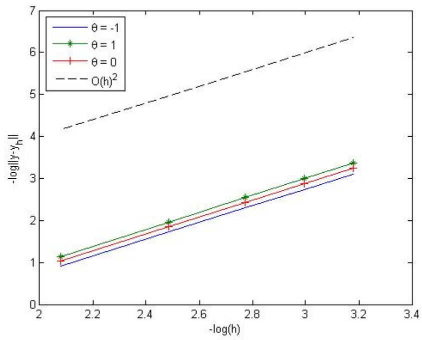
The proof follows directly by applying (2.25) and estimates of the error $\|u - u_h\|_{0,\Omega}$ and $\|y - y_h\|_{0,\Omega}$. □

2.4 Numerical experiments

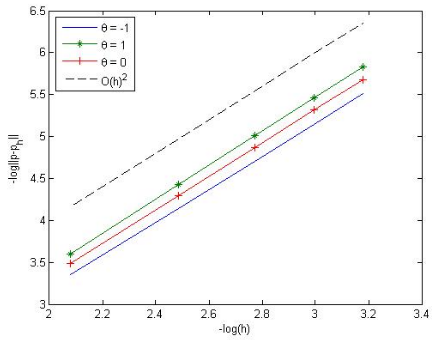
In this Section, we support our theoretical convergence results for the control, state and costate errors numerically. We stress that even the theoretical results hold true for any values of $\theta \in [-1, 1]$; but in particular, for the numerical experiments we take θ = -1,0,1, as these three values of θ have its on advantages and disadvantages over each other, for



(a) Convergence rate of $e_0(u)$.



(b) Convergence rate of $e_0(y)$.



(c) Convergence rate of $e_0(p)$.

Figure 2.3: The convergence behaviour of control, state and costate errors in L^2-norm with variational discretization approach for $\theta = -1$, $\theta = 1$, and $\theta = 0$.

more details, kindly see ([5], [40]) and references therein. To this end, we consider the following example.

Example 2.4.1. Let us consider the optimal control problem (2.1)-(2.2) with known exact solution on $\Omega = (0,1) \times (0,1)$.

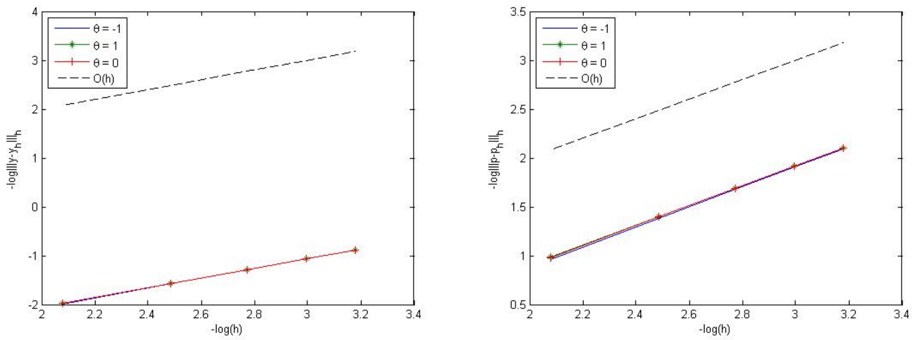
$$\min_{u \in U_{ad}} \frac{1}{2} \|y - y_d\|_{0,\Omega}^2 + \frac{\lambda}{2} \|u\|_{0,\Omega}^2$$

subject to

$$-\Delta y = u + f \quad \text{in} \quad \Omega,$$
$$y = 0 \quad \text{on} \quad \partial\Omega,$$

The given data are as follows. The exact state is $y(x_1, x_2) = 2\pi^2 \sin(\pi x_1) \sin(\pi x_2)$, costate is $p(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2)$ and control is $u(x_1, x_2) = max(-15, min(15, \frac{-1}{\lambda} p(x_1, x_2)))$ with control cost $\lambda = 0.25$. The source term f and desired state y_d are constructed respectively, as :

$f(x_1, x_2) = -\Delta y(x_1, x_2) - u(x_1, x_2)$ and $y_d(x_1, x_2) = y(x_1, x_2) + \Delta p(x_1, x_2)$.



(a) Convergence rate of $e_1(y)$. (b) Convergence rate of $e_1(p)$.

Figure 2.4: The convergence behaviour of control, state and costate errors in broken H^1-norm with variational discretization approach for $\theta = -1, \theta = 1$, and $\theta = 0$.

The state y and the costate p are discretized by piecewise linear discontinuous finite volume methods, whereas for the approximation of control u we consider three different discretization approaches: variational discretization, piecewise linear and piecewise constant discretization. The optimal control problem is solved by projected gradient method [42]. The method is formulated in Algorithm 1.

We compute the state and costate errors in L^2 and mesh dependent norm $|||\cdot|||_h$ and the control error in L^2-norm on a family of nested primal and dual triangulations of Ω.

For simplicity, we denote the errors for optimal control and the associated state and costate in L^2-norm and the corresponding observed rates by

$$\left.\begin{aligned} e_0(u) &= \|u - u_h\|_{0,\Omega}, \quad r_0(u) = \frac{log(e_0(u)/\hat{e}_0(u))}{log(h/\hat{h})}, \\ e_0(y) &= \|y - y_h\|_{0,\Omega}, \quad r_0(y) = \frac{log(e_0(y)/\hat{e}_0(y))}{log(h/\hat{h})}, \\ e_0(p) &= \|p - p_h\|_{0,\Omega}, \quad r_0(p) = \frac{log(e_0(p)/\hat{e}_0(p))}{log(h/\hat{h})}. \end{aligned}\right\} \tag{2.54}$$

Algorithm 1 Projected gradient algorithm

1: Initialization: choose u_0 and set $n = 0$, tol > 0.

2: Find $(y_n, p_n) \in H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$\left.\begin{aligned} -\nabla \cdot (\mathcal{A}\nabla y_n) &= \mathcal{B}u_n + f, \quad \text{in} \quad \Omega, \\ -\nabla \cdot (\mathcal{A}\nabla p_n) &= y_n - y_d, \quad \text{in} \quad \Omega. \end{aligned}\right\}$$

3: (Direction Search) $v_n := -(\mathcal{B}^* p_h + \lambda u_n)$

4: Set $u_{n+1} = P_{[u_a,u_b]}(u_n + \rho v_n)$ (with $\rho = \frac{1}{\lambda}$).

5: Compute the error: $E_{n+1} = |u_{n+1} - u_n|$.

6: If $E_{n+1} \leq$ tol, STOP; else go to Step 2.

Similarly, we denote the state and costate errors in discrete H^1-norm by

$$\left.\begin{aligned} e_1(y) &= \|\!|y - y_h|\!\|_h\,, \quad r_1(y) = \frac{log(e_1(y)/\hat{e}_1(y))}{log(h/\hat{h})}, \\ e_1(p) &= \|\!|p - p_h|\!\|_h\,, \quad r_1(p) = \frac{log(e_1(p)/\hat{e}_1(p))}{log(h/\hat{h})}. \end{aligned}\right\} \tag{2.55}$$

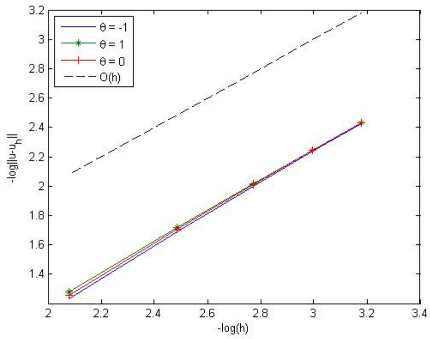
Here, e and $\hat{e}$ represents respectively, the numerical errors on two consecutive meshes of length h and $\hat{h}$. We choose the penalty parameters $\alpha = 10$ and $\beta = 1$ in the implementation.

With variational discretization approach, the numerical results of DFV approximations state, costate and control errors are listed in Table 2.1. Figure 2.3 depicts the convergence behaviour of control, state and costate errors in L^2-norm, whereas in Figure 2.4 the convergence of state and costate errors in discrete H^1-norm is shown.

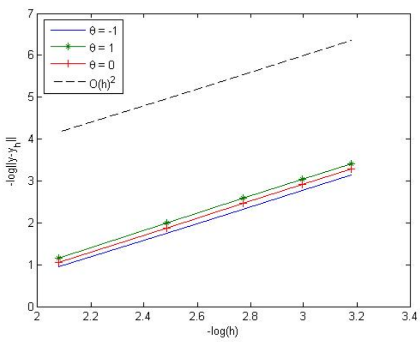
h	$e_0(y)$	$r_0(y)$	$e_1(y)$	$r_1(y)$	$e_0(p)$	$r_0(p)$	$e_1(p)$	$r_1(p)$	$e_0(u)$	$r_0(u)$
$\theta = -1$ (SIPG)										
0.1250	0.3991	-	7.2872	-	0.0350	-	0.3838	-	0.0834	-
0.0833	0.1790	1.9768	4.8475	1.0053	0.0159	1.9411	0.2505	1.0526	0.0383	1.9168
0.0625	0.1009	1.9914	3.6286	1.0066	0.0090	1.9718	0.1860	1.0335	0.0218	1.9585
0.0500	0.0646	1.9971	2.8989	1.0062	0.0058	1.9842	0.1480	1.0233	0.0140	1.9756
0.0416	0.0449	1.9997	2.4133	1.0055	0.0040	1.9903	0.1230	1.0173	0.0097	1.9843
$\theta = 1$ (NIPG)										
0.1250	0.3235	-	7.2109	-	0.0272	-	0.3722	-	0.0513	-
0.0833	0.1408	2.0516	4.8165	0.9952	0.0119	2.0267	0.2461	1.0200	0.0222	2.0639
0.0625	0.0782	2.0442	3.6125	0.9999	0.0066	2.0307	0.1840	1.0119	0.0123	2.0569
0.0500	0.0496	2.0375	2.8891	1.0013	0.0042	2.0286	0.1469	1.0083	0.0077	2.0480
0.0416	0.0342	2.0324	2.4068	1.0018	0.0029	2.0258	0.1222	1.0064	0.0053	2.0409
$\theta = 0$ (IIPG)										
0.1250	0.3561	-	7.2408	-	0.0306	-	0.3769	-	0.0653	-
0.0833	0.1571	2.0176	4.8283	0.9994	0.0137	1.9861	0.2461	1.0337	0.0918	1.9880
0.0625	0.0878	2.0203	3.6186	1.0025	0.0077	2.0027	0.1840	1.0205	0.0163	2.0066
0.0500	0.0560	2.0193	2.8928	1.0032	0.0049	2.0075	0.1473	1.0141	0.0104	2.0114
0.0416	0.0387	2.0176	2.4092	1.0032	0.0034	2.0089	0.1225	1.0106	0.0072	2.0125

Table 2.1: The computed errors for state, costate and control variables using DFV scheme with variational discretization of control variable on a sequence of uniformly refined partition of $\Omega = (0, 1)^2$.

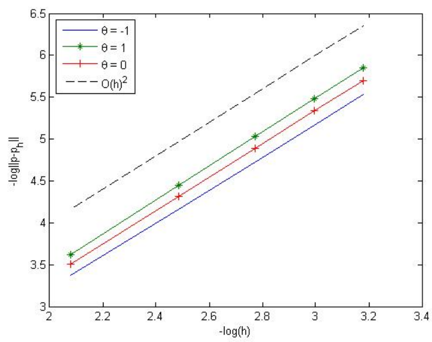
As expected, we observe a convergence of approximate $\mathcal{O}(h^2)$ for $e_0(u), e_0(y)$ and $e_0(p)$, an $\mathcal{O}(h)$ for $e_1(y)$ and $e_1(p)$.



(a) Convergence rate of $e_0(u)$.

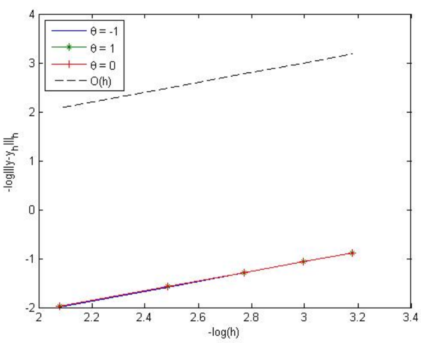


(b) Convergence rate of $e_0(y)$.

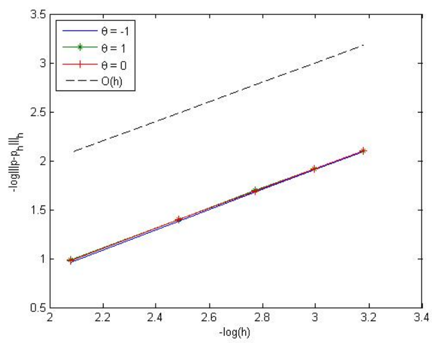


(c) Convergence rate of $e_0(p)$.

Figure 2.5: The convergence behaviour of control, state and costate errors in L^2-norm with piecewise constant control discretization for $\theta = -1$, $\theta = 1$, and $\theta = 0$.



(a) Convergence rate of $e_1(y)$.



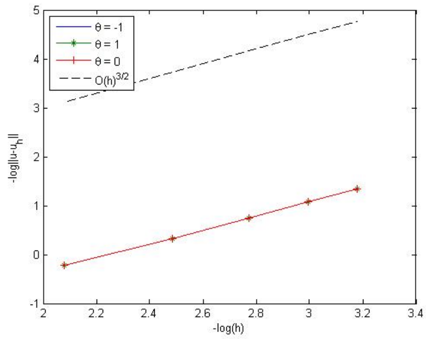
(b) Convergence rate of $e_1(p)$.

Figure 2.6: The convergence behaviour of control, state and costate errors in broken H^1-norm with piecewise constant control discretization for $\theta = -1$, $\theta = 1$, and $\theta = 0$.

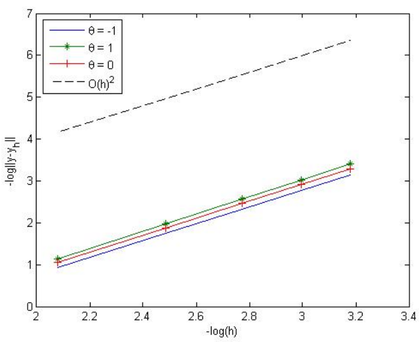
By results collected in Table 2.2, we state that the expected behaviour of errors is achieved with piecewise constant control discretization. The convergence of $\mathcal{O}(h)$ for $e_0(u)$ and $\mathcal{O}(h^2)$ for $e_0(y), e_0(p)$ is shown in Figure 2.5. Also an order h for state and costate errors in broken H^1-norm is shown which matches our theoretical results.

h	$e_0(y)$	$r_0(y)$	$e_1(y)$	$r_1(y)$	$e_0(p)$	$r_0(p)$	$e_1(p)$	$r_1(p)$	$e_0(u)$	$r_0(u)$
$\theta = -1$ (SIPG)										
0.1250	0.3864	-	7.2878	-	0.0343	-	0.3829	-	0.2919	-
0.0833	0.1731	1.9807	4.8504	1.0053	0.0156	1.9436	0.2502	1.0498	0.1843	1.1342
0.0625	0.0975	1.9937	3.6163	1.0066	0.0088	1.9731	0.1859	1.0318	0.1351	1.0781
0.0500	0.0624	1.9986	2.9015	1.0062	0.0056	1.9850	0.1480	1.0221	0.1069	1.0499
0.0416	0.0433	2.0008	2.4156	1.0055	0.0039	1.9909	0.1229	1.0165	0.0885	1.0343
$\theta = 1$ (NIPG)										
0.1250	0.3143	-	7.2170	-	0.0267	-	0.3718	-	0.2791	-
0.0833	0.1366	2.0554	4.8213	0.9952	0.0117	2.0293	0.2460	1.0183	0.1796	1.0861
0.0625	0.0758	2.0465	3.6163	0.9999	0.0065	2.0323	0.1839	1.0110	0.1330	1.0444
0.0500	0.0480	2.0392	2.8922	1.0013	0.0041	2.0297	0.1468	1.0078	0.1058	1.0266
0.0416	0.0331	2.0337	2.4094	1.0018	0.0028	2.0266	0.1222	1.0060	0.0878	1.0176
$\theta = 0$ (IIPG)										
0.1250	0.3454	-	7.2448	-	0.0300	-	0.3762	-	0.2844	-
0.0833	0.1522	2.0214	4.8324	0.9994	0.0134	1.9886	0.2476	1.0315	0.1815	1.1070
0.0625	0.0850	2.0226	3.6220	1.0025	0.0075	2.0041	0.1847	1.0193	0.1339	1.0584
0.0500	0.0541	2.0209	2.8957	1.0032	0.0048	2.0084	0.1473	1.0134	0.1062	1.0360
0.0416	0.0374	2.0189	2.4117	1.0032	0.0033	2.0095	0.1225	1.0101	0.0881	1.0243

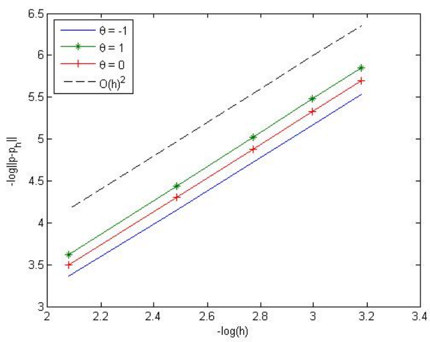
Table 2.2: The computed errors for state, costate and control variables using DFV scheme with piecewise constant discretization of control variable on a sequence of uniformly refined partition of $\Omega = (0, 1)^2$.



(a) Convergence rate of $e_0(u)$.

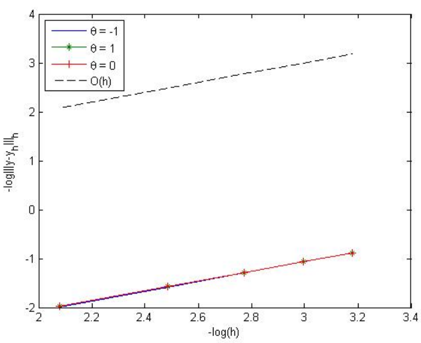


(b) Convergence rate of $e_0(y)$.

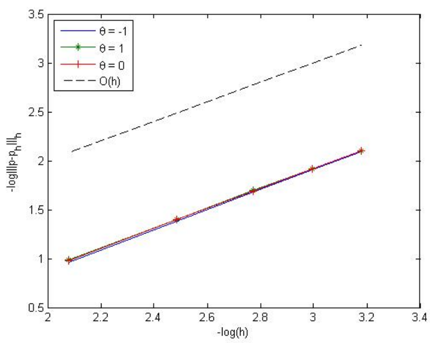


(c) Convergence rate of $e_0(p)$.

Figure 2.7: The convergence behaviour of control, state and costate errors in L^2-norm with piecewise linear control discretization for $\theta = -1$, $\theta = 1$, and $\theta = 0$.



(a) Convergence rate of $e_1(y)$.



(b) Convergence rate of $e_1(p)$.

Figure 2.8: The convergence behaviour of control, state and costate errors in broken H^1-norm with piecewise linear control discretization for $\theta = -1$, $\theta = 1$, and $\theta = 0$.

For piecwise linear control discretization the experimental results are shown in Table 2.3, Figure 2.7 and Figure 2.8. The convergence of approximate $\mathcal{O}(h^{3/2})$ for $e_0(u)$ and other expected convergence orders for state and costate errors in respective norms is achieved. An average iteration count through all (refinement levels and different control discretization) is five to achieve tolerance 10^{-6}.

h	$e_0(y)$	$r_0(y)$	$e_1(y)$	$r_1(y)$	$e_0(p)$	$r_0(p)$	$e_1(p)$	$r_1(p)$	$e_0(u)$	$r_0(u)$
$\theta = -1$ (SIPG)										
0.1250	0.3884	-	7.2888	-	0.0345	-	0.3831	-	0.2177	-
0.0833	0.1738	1.9822	4.8507	1.0043	0.0156	1.9445	0.2502	1.0503	0.1229	1.4104
0.0625	0.0979	1.9950	3.6317	1.0060	0.0088	1.9739	0.1859	1.0320	0.0806	1.4648
0.0500	0.0626	1.9998	2.9016	1.0057	0.0057	1.9857	0.1480	1.0223	0.0579	1.4853
0.0416	0.0435	2.0018	2.4157	1.0052	0.0039	1.9915	0.1229	1.0166	0.0440	1.4946
$\theta = 1$ (NIPG)										
0.1250	0.3160	-	7.2174	-	0.0268	-	0.3718	-	0.2154	-
0.0833	0.1372	2.0572	4.8214	0.9949	0.0117	2.0306	0.2460	1.0187	0.1211	1.0861
0.0625	0.0761	2.0482	3.6163	0.9997	0.0065	2.0334	0.1839	1.0112	0.0794	1.0444
0.0500	0.0482	2.0406	2.8922	1.0012	0.0041	2.0306	0.1468	1.0079	0.0571	1.0266
0.0416	0.0333	2.0349	2.4094	1.0017	0.0028	2.0274	0.1222	1.0061	0.0435	1.0176
$\theta = 0$ (IIPG)										
0.1250	0.3472	-	7.2454	-	0.0301	-	0.3763	-	0.2158	-
0.0833	0.1529	2.0231	4.8325	0.9988	0.0134	1.9897	0.2476	1.0319	0.1216	1.4201
0.0625	0.0854	2.0241	3.6221	1.0022	0.0075	2.0051	0.1847	1.0195	0.0798	1.4633
0.0500	0.0543	2.0222	2.8957	1.0029	0.0048	2.0092	0.1473	1.0135	0.0573	1.4799
0.0416	0.0376	2.0200	2.4118	1.0030	0.0033	2.0103	0.1225	1.0101	0.0437	1.4878

Table 2.3: The computed errors for state, costate and control variables using DFV scheme with piecewise linear discretization of control variable on a sequence of uniformly refined partition of $\Omega = (0, 1)^2$.

Semilinear elliptic optimal control problem:

In this part, we extend the analysis of discontinuous finite volume methods to the numerical approximations of the optimal control problems governed by a class of semilinear elliptic equations with control constraints. For the approximation of control variable, here also we have adopted three different methodologies: variational discretization, piecewise constant and piecewise linear discretization, while the approximation of state and costate variables are based on discontinuous piecewise linear polynomials. Optimal *a priori* error estimates in suitable natural norms for state, costate and control variables are derived. Moreover, numerical experiments are presented to support the derived theoretical results.

2.5 Introduction (Semilinear)

Let us consider the following optimization problem governed by semilinear elliptic equations with control u and state $y = y(u)$

$$\min_{u \in U_{ad}} J(u) := \frac{1}{2} \|y - y_d\|_{0,\Omega}^2 + \frac{\lambda}{2} \|u\|_{0,\Omega}^2 \,, \tag{2.56}$$

subject to

$$\left.\begin{aligned} -\nabla \cdot (\mathcal{A}\nabla y) + \varphi(y) = \mathcal{B}u + f, \quad \text{in} \quad \Omega,\\ y = 0 \quad \text{on} \quad \partial\Omega. \end{aligned}\right\} \tag{2.57}$$

Here, the set of admissible controls U_{ad} is defined by

$$U_{ad} = \{u \in \mathcal{U} = L^\infty(\Omega) : u_a \leqslant u(x) \leqslant u_b,\, a.e.\, in\, \Omega\},$$

with the bounds $u_a, u_b \in \mathbb{R}$ that fulfill $u_a < u_b$, As used before, $\lambda > 0$ denotes regularization parameter, $\mathcal{B}$ is bounded linear operator and $\mathcal{A}$ is a real valued, symmetric and uniformly positive definite matrix in Ω.

For our forthcoming analysis we make following assumptions on the given data. Let the desired state y_d and the source term $f \in L^\infty(\Omega)$ or $H^1(\Omega) \cap L^\infty(\Omega)$. As in [16, 67], we assume that the nonlinear term φ is of class C^2 , $\varphi' > 0$ and for $y = 0$ the derivatives of φ upto second order are bounded by a positive constant. Also, on bounded sets, they are uniformly Lipschitz continuous.

With the help of classical arguments (as mentioned in [79]), the existence of at least one optimal control u with associated state $y = y(u)$ of problem (2.56)-(2.57) can be proved. Due to nonlinearity of state equation the optimization problem is non-convex and hence the solutions may not be unique without imposing additional assumptions. Theoretically, arbitrarily many global and local minima are possible. Therefore, we will assume a locally optimal reference control (see [16, 79]) of problem (2.56)-(2.57) which satisfies first order necessary and second order sufficient optimality conditions.

Definition 2.5.1. *A control $u \in U_{ad}$ is said to be a local solution of* (2.56)-(2.57) *in the sense of L^2, if there exists some $\varepsilon > 0$ such that the following holds true*

$$J(u) \leq J(\tilde{u}), \quad \forall \tilde{u} \in U_{ad} \text{ with } \|\tilde{u} - u\|_{0,\Omega} \leq \varepsilon.$$

The local solution u of (2.56)-(2.57) in the sense of Definition 2.5.1 satisfies the following first order necessary optimality condition

$$J'(u)(\tilde{u} - u) \geq 0, \quad \forall \tilde{u} \in U_{ad},$$

and can be further formulated with the help of the following variational inequality

$$(\lambda u + \mathcal{B}^* p, \tilde{u} - u) \geq 0, \quad \forall \tilde{u} \in U_{ad}. \tag{2.58}$$

Here, $\mathcal{B}^*$ is the adjoint of operator $\mathcal{B}$, $p = p(u)$ is called the *adjoint state* (or *costate*) associated with local control u and solves the *adjoint state equation*

$$\left.\begin{aligned} -\nabla \cdot (\mathcal{A}\, \nabla p) + \varphi'(y)p = y - y_d, \quad \text{in} \quad \Omega, \\ p = 0, \quad \text{on} \quad \partial\Omega. \end{aligned}\right\}$$

We assume that a local optimal solution $u \in U_{ad}$ of (2.56)-(2.57) satisfies the following second order sufficient optimality condition (for details, see [54, 79, 20]).

$$\text{There exists a constant } C > 0 \text{ such that } J''(u)(\tilde{u}, \tilde{u}) \geq C \|\tilde{u}\|_{0,\Omega}^2, \ \forall \tilde{u} \in \mathcal{U}. \tag{2.59}$$

As far as numerical approximation of these problems is concerned, FE methods have been employed (see e.g.[16]). Keeping in mind the computational advantages of FV and DG schemes in regard of applications of semilinear elliptic control problems, the main objective of this Chapter is to study convergence analysis of DFV methods for the approximation of semilinear elliptic optimal control problems with distributed control.

We would like to mention that to the best of our knowledge, this is the first result addressing DFV approximations of semilinear elliptic problems. In this Chapter, we focus on *optimize-then-discretize* approach combined with DFV for the numerical treatment of semilinear elliptic optimal control problems, and optimal error estimates for all unknown variables are established. Furthermore, numerical examples are presented to illustrate the performance of the proposed scheme and to validate the desired theoretical results.

We have arranged the contents of this part as follows. Section 2.5 deals with the statement of the model problem, the corresponding optimality conditions and related work. Section 2.6 is devoted to the DFV formulation of the considered optimal control problem. In Section 2.7 we develop our main results on *a priori* error estimates for the three different control discretizations (variational, piecewise constant and linear) in suitable norms. Finally, in Section 2.8, we present numerical experiments to verify the theoretical results and judge the performance of the method.

2.6 Discretization

In this Section we apply DFV methods directly for the optimal control problem (2.56)-(2.57). For the discretization of control we will consider three different techniques: variational discretization, piecewise linear and constant discretization defined in Section 2.2.2.

Discontinuous finite volume discretizations

At first, we present discontinuous finite volume schemes described in Section 2.2.1 for the discretization of state equation (2.57).

On testing the state equation (2.57) against γv_h, integrating by parts over control volumes, applying Gauss divergence theorem and following the arguments used in [49, 50, 81], we end up with the following standard DFV formulation corresponding to state equation : For a given u, find $y_h(u) \in V_h$ such that

$$A_h(y_h(u), v_h) + (\varphi(y_h(u)), \gamma v_h) = (\mathcal{B}u + f, \gamma v_h), \quad \forall v_h \in V_h, \tag{2.60}$$

As $[\![y]\!] = 0$ and $[\![\gamma y]\!] = 0$, the scheme in (2.60) is consistent, i.e. the solution y satisfies

$$A_h(y, v_h) + (\varphi(y), \gamma v_h) = (\mathcal{B}u + f, \gamma v_h) \quad \forall v_h \in V_h. \tag{2.61}$$

Now, we define a discontinuous interpolation operator $I_h : C(\Omega) \longrightarrow V_h$ by

$$(I_h v)|_K := \sum_{i=1}^{3} v_i \phi_i, \quad K \in \mathcal{T}_h,$$

where $\{\phi_i\}_{i=1}^3$ be the standard local basis functions for the finite dimensional space V_h associated with triangle K and $v_i' s$ are the nodal values of function v on triangle $K \in \mathcal{T}_h$. The interpolant I_h has the following approximation properties

$$|v - I_h v|_{s,K} \le C h_K^{2-s} \|v\|_{2,K}, \quad \forall K \in \mathcal{T}_h, \quad s = 0, 1.$$

From (2.60) and (2.61), we obtain the following property

$$A_h(y - y_h(u), v_h) + (\varphi(y) - \varphi(y_h(u)), \gamma v_h) = 0 \quad \forall v_h \in V_h. \tag{2.62}$$

We will use the following discrete Sobolev inequality in our forthcoming analysis (for a proof we refer to [10, Lemma 5.2])

$$\|v_h\|_{0,\infty} \le C |\ln h|^{1/2} |\!|\!| v_h |\!|\!|_h, \qquad \forall v_h \in V_h, \tag{2.63}$$

where the norm $|\!|\!| \cdot |\!|\!|_h$ is already defined in (2.16).

Using the coercivity of the bilinear form $A_h(\cdot, \cdot)$ and monotonicity of φ (for details we refer to proof of [87, Lemma 4.1]), it is easy to verify the following result

$$A_h(v_h, v_h) + (\varphi'(y)v_h, \gamma v_h) \geq C \, |||v_h|||_h^2 \,. \tag{2.64}$$

On applying the DFV scheme to discretize the state and costate equations directly, the discrete formulation of semilinear elliptic optimal control problem (2.56)-(2.57) is given by : Find $(y_h, p_h, u_h) \in V_h \times V_h \times U_{h,ad}$ such that

$$A_h(y_h, v_h) + (\varphi(y_h), \gamma v_h) = (\mathcal{B}u_h + f, \gamma v_h), \quad \forall v_h \in V_h, \tag{2.65}$$

$$A_h(p_h, q_h) + (\varphi'(y_h)p_h, \gamma q_h) = (y_h - y_d, \gamma q_h), \quad \forall q_h \in V_h, \tag{2.66}$$

$$(\lambda u_h + \mathcal{B}^* p_h, \tilde{u}_h - u_h) \geq 0, \quad \forall \tilde{u}_h \in U_{h,ad}. \tag{2.67}$$

2.7 Error estimates

In this Section, we analyze the discretization error for a fixed local optimal reference solution of the problem (2.56)-(2.57) which satisfies the standard first order necessary and second order sufficient optimality conditions. We will derive the convergence results for variational discretization, piecewise linear and constant control discretization techniques. To start with, we consider the following auxiliary equations.

For a given arbitrary $\tilde{u} \in L^2(\Omega)$ and $\tilde{y} = y(\tilde{u}) \in H_0^1(\Omega)$, let $y_h(\tilde{u})$ and $p_h(\tilde{y})$ be the solutions of

$$A_h(y_h(\tilde{u}), v_h) + (\varphi(y_h(\tilde{u})), \gamma v_h) = (\mathcal{B}\tilde{u} + f, \gamma v_h), \ \forall v_h \in V_h, \tag{2.68}$$

and

$$A_h(p_h(\tilde{y}), q_h) + (\varphi'(\tilde{y})p_h(\tilde{y}), \gamma q_h) = (\tilde{y} - y_d, \gamma q_h), \ \forall q_h \in V_h, \tag{2.69}$$

respectively. For simplicity, we will make use of the following notations: $y_h = y_h(u_h)$, $p_h = p_h(y_h)$ and $p_h(\tilde{u}) = p_h(y_h(\tilde{u}))$. We note that all controls $u, \tilde{u}, u_h, \tilde{u}_h$ are contained in U_{ad}, and therefore, they are uniformly bounded. The same holds true for all associated states and adjoint states. Now, we prove the following error estimates for the above auxiliary discrete state and adjoint equations for a given $\tilde{u} = u$ and $\tilde{y} = y$ in broken

H^1-norm and L^2-norm.

Theorem 2.7.1. *Let $y_h(u)$ and $p_h(y)$ be the solutions of* (2.68) *and* (2.69)*, respectively. Then there exists a positive constant C independent of h such that the following error estimates hold.*

$$\left\| y - y_h(u) \right\|_h + \left\| p - p_h(y) \right\|_h \leq Ch, \tag{2.70}$$

$$\|y - y_h(u)\|_{0,\Omega} + \|p - p_h(y)\|_{0,\Omega} \leq Ch^2. \tag{2.71}$$

Proof. Let $y - y_h(u) = (y - I_h y) + (I_h y - y_h(u))$, then the triangle inequality implies

$$\left\| y - y_h(u) \right\|_h \leq \left\| y - I_h y \right\|_h + \left\| I_h y - y_h(u) \right\|_h. \tag{2.72}$$

By using the definition 2.16 of $\left\| \cdot \right\|_h$, we can write

$$\left\| y - I_h y \right\|_h^2 = |y - I_h y|_{1,h}^2 + \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \int_e [\![y - I_h y]\!]^2 \, \mathrm{d}s. \tag{2.73}$$

Using the trace inequality and approximation property of I_h in (2.73) we can obtain

$$\left\| y - I_h y \right\|_h \leq Ch \|y\|_{2,\Omega}. \tag{2.74}$$

In order to estimate the second term of (2.72), we utilize the result (2.64) and the property (2.62) to get

$$\begin{aligned} C \left\| I_h y - y_h(u) \right\|^2 \leq & A_h(I_h y - y_h(u), I_h y - y_h(u)) + \sum_{K \in \mathcal{T}_h} (\varphi'(y)(I_h y - y_h(u)), \gamma(I_h y \\ & - y_h(u)))_K \\ = & A_h(I_h y - y, I_h y - y_h(u)) + \sum_{K \in \mathcal{T}_h} [(\varphi'(y)(I_h y - y_h(u)), \gamma(I_h y \\ & - y_h(u)))_K + (\varphi(y_h(u)) - \varphi(y), \gamma(I_h y - y_h(u)))_K] \\ = & A_h(I_h y - y, I_h y - y_h(u)) + \sum_{K \in \mathcal{T}_h} [(\varphi'(y)(I_h y - y_h(u)), \gamma(I_h y \\ & - y_h(u)))_K + (I_h(\varphi(y_h(u)) - \varphi(y)), \gamma(I_h y - y_h(u)))_K \\ & + (\varphi(y_h(u)) - \varphi(y) - I_h(\varphi(y_h(u)) - \varphi(y)), \gamma(I_h y - y_h(u)))_K]. \end{aligned}$$

Applying Taylor's expansion (cf. [87]), the estimate (2.74) and (2.63) in the above relation, we can achieve

$$C \left\| I_h y - y_h(u) \right\|^2 \leq Ch \left\| I_h y - y_h(u) \right\|_h + C[h|\ln h|^{1/2} \left\| I_h y - y_h(u) \right\|_h$$
$$+ |\ln h| \left\| I_h y - y_h(u) \right\|_h^2] \left\| I_h y - y_h(u) \right\|_h + Ch \left\| I_h y - y_h(u) \right\|_h .$$

Further, we use the continuity argument ([37]) to obtain

$$\left\| I_h y - y_h(u) \right\|_h \leq Ch \left\| y \right\|_{2,\Omega} . \tag{2.75}$$

Inserting the relations (2.74) and (2.75) in (2.72) we can readily obtain the desired estimate

$$\left\| y - y_h(u) \right\|_h \leq Ch. \tag{2.76}$$

Analogously, we can achieve $\left\| p - p_h(y) \right\|_h \leq Ch$ and hence the desired estimate (2.70). Now, let $\tilde{\varphi}$ be a function (introduced in [67]) which is defined by

$$\tilde{\varphi}(x) = \begin{cases} \frac{\varphi(y)-\varphi(y_h(u))}{y-y_h(u)}, & \text{if} \quad y \neq y_h(u) \\ 0, & \text{else,} \end{cases} \tag{2.77}$$

where, $\|\tilde{\varphi}\|_\infty \leq c$ for a $c > 0$ due to boundedness of U_{ad}. For deriving the optimal estimates in L^2-norm, we consider the following auxiliary dual problem :

$$\begin{aligned} -\nabla \cdot \mathcal{A} \nabla z + \tilde{\varphi} z &= y - y_h(u) \quad \text{in} \quad \Omega, \\ z &= 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{2.78}$$

with the following relation

$$\|z\|_{2,\Omega} \leq C \left\| y - y_h(u) \right\|_{0,\Omega} . \tag{2.79}$$

On multiplying both sides of (2.78) by $y - y_h(u)$ and using definition (2.77) of $\tilde{\varphi}$, we can obtain

$$\begin{aligned}\|y-y_h(u)\|_{0,\Omega}^2 =& a_h(y-y_h(u),z)+(y-y_h(u),\tilde{\varphi}z)\\ =& a_h(y-y_h(u),z-z_I)+[a_h(y-y_h(u),z_I)-A_h(y-y_h(u),z_I)]\\ &+[A_h(y-y_h(u),z_I)+(\varphi(y)-\varphi(y_h(u)),z)]\,,\end{aligned}$$

where $I_h z = z_I \in V_h$ is piecewise linear interpolant of z. Using the relation (2.62) in the above equation we get

$$\begin{aligned}\|y-y_h(u)\|_{0,\Omega}^2 =& a_h(y-y_h(u),z-z_I)+[a_h(y-y_h(u),z_I)-A_h(y-y_h(u),z_I)]\\ &+[(\varphi(y_h(u))-\varphi(y),\gamma z_I)+(\varphi(y)-\varphi(y_h(u)),z)]\end{aligned} \tag{2.80}$$

Using the boundedness of bilinear form $a_h(\cdot,\cdot)$, the approximation property of I_h, the estimate (2.76) and relation (2.79), we can bound the first term of (2.80) by

$$a_h(y-y_h(u),z-z_I)\leq |||y-y_h(u)|||_h\,|||z-z_I|||_h\leq Ch^2\,\|y-y_h(u)\|_{0,\Omega}\,.$$

For the second term in (2.80), we use the results of [49, Lemma 3.1] to obtain

$$|a_h(y-y_h(u),z_I)-A_h(y-y_h(u),z_I)|\leq Ch^2\,\|z_I\|_1\leq Ch^2\,\|z\|_2\leq Ch^2\,\|y-y_h(u)\|_{0,\Omega}\,.$$

For the third term of (2.80), the approximation property of I_h and γ, Lipschitz continuity of nonlinear term φ and relation (2.79) gives

$$\begin{aligned}(\varphi(y)-\varphi(y_h(u)),z-\gamma z_I) &= (\varphi(y)-\varphi(y_h(u)),z-z_I)+(\varphi(y)-\varphi(y_h(u)),z_I-\gamma z_I)\\ &\leq Ch^2\,\|y-y_h(u)\|_{0,\Omega}^2+Ch\,\|y-y_h(u)\|_{0,\Omega}^2\,.\end{aligned}$$

The proof follows directly by inserting the bounds of all the terms in (2.80). We can also obtain $\|p-p_h(y)\|_{0,\Omega}\leq Ch^2$ by following analogous steps as above. □

Similarly, it can be proved that, for $\tilde{u}=u_h$ we have

$$\|p(u_h)-p_h(u_h)\|_{L^2(L^2)}\leq Ch^2. \tag{2.81}$$

2.7.1 Error estimates for control

With variational discretization:

Let us now develop the L^2- error estimates of control variables by following variational discretization approach. In this approach the control set is not discretized explicitly, we choose $U_h = L^2(\Omega)$ and thus $U_{h,ad} = U_{ad}$. We will utilize the coercivity of J'' to derive the results of following Theorem.

Theorem 2.7.2. *Let u be a local optimal control of the problem (2.56)-(2.57) and u_h be its approximation with variational discretization approach. Then the following convergence result is obtained*

$$\|u - u_h\|_{0,\Omega} \leq Ch^2. \tag{2.82}$$

Proof. From the discrete and continuous variational inequalities we have

$$(\lambda u_h + \mathcal{B}^* p_h, u - u_h) \geq 0 \geq (\lambda u + \mathcal{B}^* p, u - u_h). \tag{2.83}$$

The second order sufficient condition (2.59) for $u - u_h \in \mathcal{U}$ implies

$$\begin{aligned} C \|u - u_h\|_{0,\Omega}^2 &\leq J'(u)(u - u_h) - J'(u_h)(u - u_h) \\ &= (\lambda u + \mathcal{B}^* p, u - u_h) - (\lambda u_h + \mathcal{B}^* p(u_h), u - u_h) \\ &\leq (\lambda u_h + \mathcal{B}^* p_h, u - u_h) - (\lambda u_h + \mathcal{B}^* p(u_h), u - u_h), \end{aligned}$$

where, the last inequality follows from (2.83). Applying result (2.81), we can obtain the required estimate for control in L^2-norm with variational discretization approach

$$\|u - u_h\|_{0,\Omega} \leq C \|p(u_h) - p_h\|_{0,\Omega} \leq Ch^2.$$

□

With piecewise constant discretization:

First, we are going to establish an estimate for the error $\|u-u_h\|_{0,\Omega}$ when the control is discretized by piecewise constant polynomials. For this purpose we will utilize the techniques presented in [17] and the coercivity of J''. To formulate the result, we will utilize the L^2-projection operator $\Pi_0 : \mathcal{U} \longrightarrow U_h$ introduced in Section 2.3.1 with the approximation property (2.30). We note that, $\Pi_0 U_{ad} \subset U_{h,ad}$ and from the continuous and discrete variational inequalities we have the relation

$$(\lambda u_h + \mathcal{B}^* p_h, \Pi_0 u - u_h) \geq 0 \geq (\lambda u + \mathcal{B}^* p, u - u_h). \tag{2.84}$$

Theorem 2.7.3. *Let u be a local optimal control of the problem (2.56)-(2.57) and u_h be the solution of the discrete problem (2.65)-(2.67) with piecewise constant control discretization technique. Then the following convergence result holds true for a positive constant C independent of h*

$$\|u-u_h\|_{0,\Omega} \leq Ch.$$

Proof. Using (2.59) we proceed with $u - u_h \in \mathcal{U}$ to obtain

$$C\|u-u_h\|_{0,\Omega}^2 \leq (\lambda u + \mathcal{B}^* p, u-u_h) - (\lambda u_h + \mathcal{B}^* p(u_h), u-u_h).$$

Applying the condition (2.84) in the above relation implies

$$\begin{aligned} C\|u-u_h\|_{0,\Omega}^2 \leq & (\lambda u_h + \mathcal{B}^* p_h, \Pi_0 u - u_h) - (\lambda u_h + \mathcal{B}^* p(u_h), u-u_h) \\ \leq & \underbrace{(\mathcal{B}^*(p_h - p(u_h)), u-u_h)_{0,\Omega}}_{(\star)} + \underbrace{(\lambda u_h + \mathcal{B}^* p_h, \Pi_0 u - u)}_{(\star\star)}. \end{aligned} \tag{2.85}$$

To estimate the first term of (2.85), we use result (2.81) and continuity of operator $\mathcal{B}$ to get

$$(\star) \leq C\|p(u_h)-p_h\|_{0,\Omega}\|u-u_h\|_{0,\Omega} \leq Ch^2\|u-u_h\|_{0,\Omega}.$$

Using orthogonality of Π_0 and (2.30) to bound the second term of (2.85), we arrive at

$$(\star\star) = (\mathcal{B}^* p_h - \Pi_0(\mathcal{B}^* p_h), \Pi_0 u - u) \leq \|\mathcal{B}^* p_h - \Pi_0(\mathcal{B}^* p_h)\|_{0,\Omega}\|\Pi_0 u - u\|_{0,\Omega} \leq Ch^2.$$

The proof follows directly by plugging the bounds of the terms of (2.85). □

Remark 2.7.4. *For the piecewise linear control discretization, if we proceed with similar steps as above, we will end up with an $\mathcal{O}(h)$ convergence for control error in L^2-norm.*

With piecewise linear discretization:

To obtain the optimal order convergence of $\mathcal{O}(h^{3/2})$, we start by first establishing the following intermediate estimates for $\tilde{u} = u$ and $\tilde{y} = y$ which will be used in deriving the further error estimates.

Lemma 2.7.5. *There exists a constant $C > 0$ independent of h such that the following assertions holds:*

$$|||y_h(u) - y_h|||_h \leq C \|u - u_h\|_{0,\Omega}, \quad |||p_h(y) - p_h|||_h \leq C \|y - y_h\|_{0,\Omega},$$

Proof. We start analogously as in the proof of [59, Lemma 3.1] by subtracting (2.65) from (2.68). Then we have for $v_h \in V_h$

$$A_h(y_h(u) - y_h, v_h) + (\varphi(y_h(u)) - \varphi(y_h), \gamma v_h) = (\mathcal{B}(u - u_h), \gamma v_h).$$

Choosing $v_h = y_h(u) - y_h$ and denoting $y_h(u) - y_h$ by ϑ, the above equation can be rewritten as

$$A_h(\vartheta, \vartheta) = (\mathcal{B}(u - u_h), \gamma\vartheta) + (\varphi(y_h) - \varphi(y_h(u)), \gamma\vartheta).$$

Using the result (2.64) of Lemma 2.2.2 in the above equation, we have

$$\begin{aligned} |||\vartheta|||_h^2 \leq & (\mathcal{B}(u - u_h), \gamma\vartheta) + \sum_{K \in \mathcal{T}_h} [(I_h(\varphi(y_h) - \varphi(y_h(u))), \gamma\vartheta)_K + (\varphi'(y)\vartheta, \gamma\vartheta)_K] \\ & + \sum_{K \in \mathcal{T}_h} (\varphi(y_h) - \varphi(y_h(u)) - I_h(\varphi(y_h) - \varphi(y_h(u))), \gamma\vartheta)_K \\ =: & T_1 + T_2 + T_3. \end{aligned} \tag{2.86}$$

Since $\mathcal{B}$ is a continuous linear operator and γ is stable with respect to $\|\cdot\|_{0,\Omega}$, the first term T_1 of (2.86) can be bounded by using (2.17) as

$$(\mathcal{B}(u-u_h),\gamma\vartheta) \leq C\left\|u-u_h\right\|_{0,\Omega}\left\|\vartheta\right\|_{0,\Omega} \leq C\left\|u-u_h\right\|_{0,\Omega}\left\|\!\left|\vartheta\right|\!\right\|_h. \tag{2.87}$$

To bound the second term of (2.86), we make use of Taylor's expansion (see [86, 87]) and (2.17)

$$T_2 \leq C\left(\delta_1 \max_K |\vartheta| + \delta_2 \max_K |\vartheta|^2\right)\left\|\vartheta\right\|_{0,\Omega} \leq C\left(h\left\|\vartheta\right\|_{0,\infty} + \left\|\vartheta\right\|_{0,\infty}^2\right)\left\|\!\left|\vartheta\right|\!\right\|_h, \tag{2.88}$$

here, $\delta_1 = C \max_{x',x''\in K} |\varphi'(y(x')) - \varphi'(y(x''))| = \mathcal{O}(h)$ and $\delta_2 = \frac{1}{2}\varphi''(\chi) = \mathcal{O}(1)$ with $|\chi| \leq \max_{x\in\Omega} |y(x)|$. Using relation (2.63), in (2.88), we have

$$T_2 \leq C\left(h|\ln h|^{1/2}\left\|\!\left|\vartheta\right|\!\right\|_h + |\ln h|\left\|\!\left|\vartheta\right|\!\right\|_h^2\right)\left\|\!\left|\vartheta\right|\!\right\|_h. \tag{2.89}$$

The term T_3 of (2.86) is bounded by using the approximation property of I_h, Lipschitz continuity of $\varphi(\cdot)$ properties (2.10) and (2.17)

$$T_3 \leq Ch\sum_{K\in\mathcal{T}_h}\left\|\varphi(y_h)-\varphi(y_h(u))\right\|_{1,K}\left\|\gamma\vartheta\right\| \leq Ch\left\|\!\left|\vartheta\right|\!\right\|_h^2. \tag{2.90}$$

Inserting the bounds of (2.87), (2.89) and (2.90) in (2.86) and omitting the common factor $\left\|\!\left|\vartheta\right|\!\right\|_h$ we get

$$\left\|\!\left|y_h(u)-y_h\right|\!\right\|_h \leq C\left\|u-u_h\right\|_{0,\Omega} + C|\ln h|\left\|\!\left|y_h(u)-y_h\right|\!\right\|_h^2,$$

In view of $\|u-u_h\|_{0,\Omega}$ being of order h (for all discretizations), the continuity argument (see the method by Frehse-Rannacher [37]), yields the result

$$\left\|\!\left|y_h(u)-y_h\right|\!\right\|_h \leq C\left\|u-u_h\right\|_{0,\Omega}.$$

For the second result, we proceed similarly by subtracting (2.66) from (2.69), denoting $p_h(y)-p_h = \eta$ and choosing $q_h = \eta$ to get

$$A_h(\eta,\eta) = (y-y_h,\gamma\eta) + ((\varphi'(y_h)-\varphi'(y))p_h,\gamma\eta) - (\varphi'(y)\eta,\gamma\eta).$$

Using result (2.64) of Lemma 2.2.2 and Lipschitz continuity of $\varphi'(\cdot)$ in the above equation, we can obtain the relation

$$|||p_h(y) - p_h|||_h \leq C \|y - y_h\|_{0,\Omega} \, .$$

□

In deriving the optimal error estimates for $\|u - u_h\|_{0,\Omega}$ with piecewise linear control discretization, again the key point is to utilize the second order sufficient condition (2.59) for the continuous problem and the discrete and continuous variational inequalities. As in Section 2.3.1 we group each element $K \in \mathcal{T}_h$ into three sets $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2 \cup \mathcal{T}_h^3$ with $\mathcal{T}_h^i \cap \mathcal{T}_h^j = \emptyset$ for $i \neq j$ according to the value of $u(x)$ on K. These disjoint sets are defined in (2.33).

The following Theorem provides the L^2 error estimates of control variable for piecewise linear discretization.

Theorem 2.7.6. *Let u be a local optimal control of the problem (2.56)-(2.57) and u_h be the solution of the discrete problem (2.65)-(2.67) with piecewise linear control discretization technique, then the discretization error estimate*

$$\|u - u_h\|_{0,\Omega} \leq Ch^{3/2}$$

is fulfilled for a constant $C > 0$ independent of h.

Proof. The coercivity condition (2.59) for $u - u_h \in \mathcal{U}$ implies

$$C \|u - u_h\|_{0,\Omega}^2 \leq (\lambda u + \mathcal{B}^* p, u - u_h) - (\lambda u_h + \mathcal{B}^* p(u_h), u - u_h),$$

and utilizing the continuous and discrete variational inequality, we find that

$$\begin{aligned} C \|u - u_h\|_{0,\Omega}^2 \leq & (\lambda u_h + \mathcal{B}^* p_h, \tilde{u}_h - u_h) - (\lambda u_h + \mathcal{B}^* p(u_h), u - u_h) \\ \leq & (\mathcal{B}^*(p_h - p(u_h)), u - u_h) + (\lambda(u - u_h) + \mathcal{B}^*(p - p_h), u - \tilde{u}_h) \\ & + (\lambda u + \mathcal{B}^* p, \tilde{u}_h - u), \end{aligned} \tag{2.91}$$

As derived in the proof of Theorem 2.3.9, by using the definition 2.34 and projection property (2.8) together with Assumption 2.3.6, the following estimate

$$\|u - \tilde{u}_h\|_{0,\Omega} \leq Ch^{3/2} \tag{2.92}$$

can be obtained. Applying Cauchy-Schwarz inequality, property (2.17), (2.81) and results of Lemma 2.7.5 and Theorem 2.7.1 in (2.91) we get

$$\begin{aligned} \|u - u_h\|_{0,\Omega}^2 \leq & Ch^2 \|u - u_h\|_{0,\Omega} + C \|u - u_h\|_{0,\Omega} \|u - \tilde{u}_h\|_{0,\Omega} \\ & + C|(\lambda u + \mathcal{B}^* p, \tilde{u}_h - u)|. \end{aligned} \tag{2.93}$$

The proof follows by inserting (2.92) in (2.93), applying Young's inequality and Lemma 2.3.8. □

2.7.2 Error estimates for state and costate

Under variational discretization of control:

First, we would like to mention that in order to derive the L^2-error estimates for state and costate with variational discretization of control, we first split the state error as $y-y_h = y-y_h(u)+y_h(u)-y_h$ and the costate error by $p-p_h = p-p_h(y)+p_h(y)-p_h$. Then we apply triangle inequality, estimate (2.82) alongwith the results of Lemma 2.7.5 and Theorem 2.7.1 to achieve the result

$$\|y - y_h\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \leq Ch^2.$$

Under explicit control discretization:

Here also we note that that unlike the case of variational discretization approach, the derivation of optimal order L^2-error estimates for state and costate variables with the piecewise constant or linear control discretization is more involved and sophisticated. To establish $\mathcal{O}(h^2)$ convergence order, we appeal to duality arguments. Let us denote Π_h as an L^2 projection operator onto U_h. Then by triangle inequality we can have the following relation.

$$\|y - y_h\|_{0,\Omega} \leq \|y - y_h(u)\|_{0,\Omega} + \|y_h(u) - y_h(\Pi_h u)\|_{0,\Omega} + \|y_h(\Pi_h u) - y_h\|_{0,\Omega}. \tag{2.94}$$

Theorem 2.7.7. *Let u be a local reference control of the problem (2.56)-(2.57) with associated state y and costate p and let (u_h, y_h, p_h) be their discontinuous interpolated coefficient finite volume approximations. Then we have*

$$\|y - y_h\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \leq Ch^2,$$

for a positive constant C independent of h.

Proof. To begin with we define $\tilde{p}_h \in V_h$ to be the solution of auxiliary discrete dual equation

$$a_h(\tilde{p}_h, \xi) = (\xi, y_h(u) - y_h(\Pi_h u)) - (\xi, \hat{\varphi}\tilde{p}_h), \quad \forall \xi \in V_h, \tag{2.95}$$

with

$$\hat{\varphi}(x) = \begin{cases} \frac{\varphi(y_h(u)) - \varphi(y_h(\Pi_h u))}{y_h(u) - y_h(\Pi_h u)}, & \text{if} \quad y_h(u) \neq y_h(\Pi_h u)) \\ 0, & \text{else.} \end{cases}$$

Where $\|\hat{\varphi}\|_{0,\infty} \leq c$ for a $c > 0$ due to boundedness of U_{ad}, and the following result holds

$$\|\tilde{p}_h\|_h \leq C \|y_h(u) - y_h(\Pi_h u)\|_{0,\Omega}. \tag{2.96}$$

Choosing $\xi = y_h(u) - y_h(\Pi_h u)$ in (2.95) we have

$$a_h(\tilde{p}_h, y_h(u) - y_h(\Pi_h u)) = \|y_h(u) - y_h(\Pi_h u))\|^2_{0,\Omega} - (\varphi(y_h(u)) - \varphi(y_h(\Pi_h u)), \tilde{p}_h). \tag{2.97}$$

From the discrete state equation for $y_h(u)$ and $y_h(\Pi_h u)$, we have the relation

$$A_h(y_h(u) - y_h(\Pi_h u), \tilde{p}_h) = (\mathcal{B}(u - \Pi_h u), \gamma\tilde{p}_h) - (\varphi(y_h(u)) - \varphi(y_h(\Pi_h u)), \gamma\tilde{p}_h). \tag{2.98}$$

On subtracting (2.98) from (2.97) and using the definition of $\epsilon_a(\cdot,\cdot)$, we can write

$$\begin{aligned}\|y_h(u) - y_h(\Pi_h u))\|_{0,\Omega}^2 =& (\mathcal{B}(u - \Pi_h u), \gamma\tilde{p}_h) + \epsilon_a(y_h(u) - y_h(\Pi_h u), \tilde{p}_h) \\ &+ [(\varphi(y_h(u)) - \varphi(y_h(\Pi_h u)), \tilde{p}_h - \gamma\tilde{p}_h)] \\ &=: R_1 + R_2 + R_3. \end{aligned} \tag{2.99}$$

The first term R_1 of (2.99) can be bounded by using the property of projection Π_h, results of Lemma 2.2.1 and (2.96)

$$\begin{aligned} R_1 &= (\mathcal{B}(u - \Pi_h u), \gamma\tilde{p}_h - \tilde{p}_h) + (u - \Pi_h u, \mathcal{B}^*\tilde{p}_h - \Pi_h\mathcal{B}^*\tilde{p}_h) \\ &\leq Ch \|u - \Pi_h u\|_{0,\Omega} |||\tilde{p}_h|||_h \leq Ch \|u - \Pi_h u\|_{0,\Omega} \|y_h(u) - y_h(\Pi_h u)\|_{0,\Omega}. \end{aligned}$$

From the estimate of $\epsilon_a(\cdot,\cdot)$ in Lemma 2.2.2 and using (2.96), one can obtain

$$\begin{aligned} R_2 \leq Ch |||y_h(u) - y_h(\Pi_h u)|||_h |||\tilde{p}_h|||_h &\leq Ch |||y_h(u) - y_h(\Pi_h u)|||_h \|y_h(u) - y_h(\Pi_h u)\|_{0,\Omega} \\ &\leq Ch \|u - \Pi_h u\|_{0,\Omega} \|y_h(u) - y_h(\Pi_h u)\|_{0,\Omega}, \end{aligned}$$

where, we have used the relation $|||y_h(u) - y_h(\Pi_h u)|||_h \leq \|u - \Pi_h u\|_{0,\Omega}$, which is analogue of the result of Lemma 2.7.5. To estimate R_3 we will use approximation properties of γ, the Lipschitz continuity of nonlinear term $\varphi(\cdot)$ and relation (2.96)

$$\begin{aligned} R_3 \leq Ch \|y_h(u) - y_h(\Pi_h u)\|_{0,\Omega} |||\tilde{p}_h|||_h &\leq Ch |||y_h(u) - y_h(\Pi_h u)|||_h \|y_h(u) - y_h(\Pi_h u)\|_{0,\Omega} \\ &\leq Ch \|u - \Pi_h u\|_{0,\Omega} \|y_h(u) - y_h(\Pi_h u)\|_{0,\Omega}. \end{aligned}$$

Finally substituting the estimates of the terms R_1, R_2 and R_3 in (2.99) we find that

$$\|y_h(u) - y_h(\Pi_h u)\|_{0,\Omega} \leq Ch \|u - \Pi_h u\|_{0,\Omega}. \tag{2.100}$$

For the third term in (2.94), using (2.17) and using the results of Lemma 2.7.5, we can obtain

$$\|y_h(\Pi_h u) - y_h\|_{0,\Omega} \leq |||y_h(\Pi_h u) - y_h|||_h \leq C \|\Pi_h u - u_h\|_{0,\Omega}. \tag{2.101}$$

Applying condition (2.59) for $\Pi_h u - u_h \in U_{h,ad} \subset \mathcal{U}$, we find that

$$C \left\| \Pi_h u - u_h \right\|_{0,\Omega}^2 \leq (\lambda \Pi_h u + \mathcal{B}^* p(\Pi_h u), \Pi_h u - u_h) - (\lambda u_h + \mathcal{B}^* p(u_h), \Pi_h u - u_h)$$
$$\leq \lambda \left\| \Pi_h u - u_h \right\|_{0,\Omega}^2 - (\mathcal{B}^* p(u_h) - \mathcal{B}^* p(\Pi_h u), \Pi_h u - u_h). \quad (2.102)$$

The discrete variational inequality and projection property of Π_h alongwith result (2.35) implies the following relation

$$\begin{aligned}\lambda \left\| \Pi_h u - u_h \right\|_{0,\Omega}^2 =& \lambda(u - u_h, \Pi_h u - u_h) \\ \leq& (\mathcal{B}^* p_h - \mathcal{B}^* p, \Pi_h u - u_h) \\ =& (\mathcal{B}^* p_h - \mathcal{B}^* p(u_h), \Pi_h u - u_h) + (\mathcal{B}^* p(u_h) - \mathcal{B}^* p(\Pi_h u), \Pi_h u - u_h) \\ &+ (\mathcal{B}^* p(\Pi_h u) - \mathcal{B}^* p, \Pi_h u - u_h).\end{aligned}$$

Therefore, we can rewrite the above relation as

$$\lambda \left\| \Pi_h u - u_h \right\|_{0,\Omega}^2 - (\mathcal{B}^* p(u_h) - \mathcal{B}^* p(\Pi_h u), \Pi_h u - u_h)$$
$$\leq (\mathcal{B}^* p_h - \mathcal{B}^* p(u_h), \Pi_h u - u_h) + (\mathcal{B}^* p(\Pi_h u) - \mathcal{B}^* p, \Pi_h u - u_h). \quad (2.103)$$

To bound the first term of (2.103) we use the result (2.71) of Lemma 2.7.1 and continuity of operator $\mathcal{B}$ to obtain the estimate

$$(\mathcal{B}^* p_h - \mathcal{B}^* p(u_h), \Pi_h u - u_h) \leq C h^2 \left\| \Pi_h u - u_h \right\|_{0,\Omega}. \quad (2.104)$$

Decomposing the second term of (2.103) as

$$\begin{aligned}(\mathcal{B}^* p(\Pi_h u) - \mathcal{B}^* p, \Pi_h u - u_h) =& (\mathcal{B}^* p(\Pi_h u) - \mathcal{B}^* p_h(\Pi_h u), \Pi_h u - u_h) \\ &+ (\mathcal{B}^* p_h(\Pi_h u) - \mathcal{B}^* p_h(u), \Pi_h u - u_h) \\ &+ (\mathcal{B}^* p_h(u) - \mathcal{B}^* p, \Pi_h u - u_h).\end{aligned}$$

Applying the results of Lemma 2.7.1 in the above relation we can obtain the bound

$$\begin{aligned}
&(\mathcal{B}^*p(\Pi_h u) - \mathcal{B}^*p, \Pi_h u - u_h) \\
&\leq Ch^2 \|\Pi_h u - u_h\|_{0,\Omega} + \|p_h(y_h(\Pi_h u)) - p_h(y_h(u)\|_{0,\Omega} \|\Pi_h u - u_h\|_{0,\Omega} \\
&\leq Ch^2 \|\Pi_h u - u_h\|_{0,\Omega} + \|y_h(\Pi_h u) - y_h(u)\|_{0,\Omega} \|\Pi_h u - u_h\|_{0,\Omega} \\
&\leq Ch^2 \|\Pi_h u - u_h\|_{0,\Omega} + Ch \|u - \Pi_h u\|_{0,\Omega} \|\Pi_h u - u_h\|_{0,\Omega}, \qquad (2.105)
\end{aligned}$$

where, the last inequality follows from the proof of Lemma 2.7.5 and estimate (2.100). Now, we use the estimates of (2.103), (2.104) and (2.105) in (2.102) and insert it in (2.101) to get

$$\|y_h(\Pi_h u) - y_h\|_{0,\Omega} \leq Ch \|u - \Pi_h u\|_{0,\Omega}. \qquad (2.106)$$

Inserting the terms (2.100) and (2.106) in (2.94), using the L^2 estimates of Lemma 2.7.1 and approximation properties of $\Pi_h u$, the optimal order of convergence for state error with piecewise constant linear discretization of control can be obtained, i.e.,

$$\|y - y_h\|_{0,\Omega} = \mathcal{O}(h^2). \qquad (2.107)$$

Now, we can split the costate error as $p - p_h = p - p_h(y) + p_h(y) - p_h$ and apply triangle inequality along with the estimates of Lemma 2.7.5, Lemma 2.7.1 and above result (2.107) to obtain

$$\begin{aligned}
\|p - p_h\|_{0,\Omega} \leq \|p - p_h(y)\|_{0,\Omega} + |||p_h(y) - p_h|||_h &\leq \|p - p_h(y)\|_{0,\Omega} + \|y - y_h\|_{0,\Omega} \\
&= \mathcal{O}(h^2).
\end{aligned}$$

□

In mesh dependent norm:

To derive the error bounds for state and costate error in energy norm, we start by decomposing the state and costate errors as $y - y_h = y - y_h(u) + y_h(u) - y_h$ and $p - p_h = p - p_h(y) + p_h(y) - p_h$, respectively. Then on applying triangle inequality and estimates of Lemma 2.7.5 we achieve

$$\|\|y - y_h\|\|_h \leq \|\|y - y_h(u)\|\|_h + C\,\|u - u_h\|_{0,\Omega} \tag{2.108}$$

$$\|\|p - p_h\|\|_h \leq \|\|p - p_h(y)\|\|_h + C\,\|y - y_h\|_{0,\Omega}\,. \tag{2.109}$$

Now let us formulate the estimates of state and costate error in mesh dependent norm by the following Theorem.

Theorem 2.7.8. *Let u be a fixed local control of the problem (2.56)-(2.57) with associated state y and costate p and let (u_h, y_h, p_h) be their approximations with discontinuous interpolated finite volume method. Then we have*

$$\|\|y - y_h\|\|_h + \|\|p - p_h\|\|_h \leq Ch.$$

The proof follows directly by the application of result (2.70) of Theorem 2.7.1 in (2.108) and (2.109).

2.8 Numerical Experiments

In this Section, present two numerical examples to illustrate the performance of the proposed method and to confirm the theoretical results of Section 2.7.

Implementation aspects

In general, the resulting nonlinear algebraic system is solved by Newton iteration method. At each iteration, we need to compute the Jacobian matrix which involves derivative, and this is computationally expensive. In order to overcome this difficulty, we will use the idea of interpolated coefficients (introduced in [85, 86, 87, 88]) for our semilinear elliptic optimal control problem (2.56)-(2.57).

Then on utilizing the idea of interpolated coefficients, the discrete state equation (2.65) can be reformulated as

$$A_h(y_h, v_h) + (I_h\varphi(y_h), \gamma v_h) = (\mathcal{B}u_h + f, \gamma v_h). \tag{2.110}$$

Let $\{\Phi_i\}_{i=1}^{N_d}$, $\{\Phi_i^*\}_{i=1}^{N_d}$ and $\{\Psi_i\}_{i=1}^{M_d}$ be the basis functions for V_h, V_h^* and U_h, respectively. If define the matrix blocks and vectors

$$\mathbb{A} = [A_h(\Phi_i, \Phi_j)]_{1\le i,j\le N_d} = (a_{ij})_{N_d\times N_d}, \quad \mathbb{M} = [(\Phi_i, \Phi_j^*)]_{1\le i,j\le N_d} = (m_{ij})_{N_d\times N_d}$$
$$\mathbb{G} = [A_h(\Phi_i^*, \Psi_j)]_{1\le i\le N_d, 1\le j\le M} = (g_{ij})_{N_d\times M_d}, \quad \mathbb{F} = [(f, \Phi_j^*)]_{1\le j\le N_d} = (f_j)_{N_d\times 1}.$$

Then the scheme in (2.110) leads to a nonlinear system of equations

$$\mathbb{A}Y + \mathbb{M}\varphi(Y) = \mathbb{G}U + \mathbb{F}. \tag{2.111}$$

where Y and U are coefficients in the expansion of y_h and u_h, respectively. The resulting non-linear system of equations is solved by applying Newton method. A direct computation shows that Jacobian matrix is $\mathbb{J} = \mathbb{A} + \mathbb{M}_{\varphi'}$ where $\mathbb{M}_{\varphi'}$ is the mass matrix defined as $\mathbb{M}_{\varphi'} = \int\limits_{\Omega} \frac{\partial\varphi}{\partial y}\Phi_i\Phi_j^* = \{m_{ij}\varphi'(Y_j)\}_{N\times N}$. The Jacobian matrix can be obtained by simply multiplying m_{ij} by $\varphi'(Y_j)$. We observe that the computation cost is reduced greatly as the Jacobian matrix is computed in a simple way as the derivative of nonlinear term involves direct multiplication with mass matrix and Jacobian matrix is updated once in each iteration of Newton method.

We assess the accuracy of the method by considering the following example where the exact solution of the undiscretized optimal control problem is known.

Example 2.8.1. Let us experiment for the following semilinear elliptic optimal control problem

$$\min_{u\in U_{ad}} \frac{1}{2}\|y - y_d\|_{0,\Omega}^2 + \frac{1}{2}\|u\|_{0,\Omega}^2$$

subject to

$$-\Delta y + y^3 = u + f \quad \text{in} \quad \Omega,$$
$$y = 0 \quad \text{on} \quad \partial\Omega,$$

where the source function f and the desired state y_d are given by $f(x_1, x_2) = -2(x_2^2 - x_2 + x_1^2 - x_1) + x_1^3(x_1-1)^3 x_2(x_2-1)^3 - u(x_1, x_2)$ and $y_d(x_1, x_2) = x_1(x_1-1)x_2(x_2-1) + 2(x_2^2 - x_2 + x_1^2 - x_1) - 3x_1^3(x_1-1)^3 x_2(x_2-1)^3$, respectively. The optimal control is defined by $u(x_1, x_2) = max(-0.1, min(0.1, -p(x_1, x_2)))$ with the associated optimal state $y(x_1, x_2) = x_1(x_1 - 1)x_2(x_2 - 1)$ and the optimal costate $p(x_1, x_2) = x_1(x_1 - 1)x_2(x_2 - 1)$. The domain Ω is the unit square $(0, 1) \times (0, 1)$.

h	$e_0(y)$	$r_0(y)$	$e_1(y)$	$r_1(y)$	$e_0(p)$	$r_0(p)$	$e_1(p)$	$r_1(p)$	$e_0(u)$	$r_0(u)$
Variational discretization approach										
	1.0e-002*				1.0e-002*				1.0e-003*	
0.1250	0.1458	-	0.0259	-	0.1350	-	0.0260	-	0.2797	-
0.0833	0.0650	1.9903	0.0172	1.0107	0.0601	1.9955	0.0172	1.0072	0.1288	1.9122
0.0625	0.0366	1.9997	0.0128	1.0097	0.0337	2.0032	0.0129	1.0076	0.0738	1.9333
0.0500	0.0234	2.0029	0.0102	1.0082	0.0215	2.0055	0.0103	1.0069	0.0478	1.9457
0.0416	0.0162	2.0041	0.0085	1.0071	0.0149	2.0062	0.0086	1.0061	0.0335	1.9541
Piecewise linear control										
	1.0e-002*				1.0e-002*				1.0e-002*	
0.1250	0.1279	-	0.0259	-	0.1341	-	0.0260	-	0.4032	-
0.0833	0.0565	2.0130	0.0173	1.0058	0.0597	1.9970	0.0172	1.0070	0.2310	1.3731
0.0625	0.0316	2.0168	0.0129	1.0068	0.0335	2.0043	0.0129	1.0075	0.1538	1.4137
0.0500	0.0201	2.0165	0.0103	1.0063	0.0214	2.0063	0.0103	1.0068	0.1117	1.4351
0.0416	0.0139	2.0154	0.0086	1.0057	0.0148	2.0068	0.0086	1.0060	0.0857	1.4482
Piecewise constant control										
	1.0e-002*				1.0e-002*				1.0e-002*	
0.1250	0.1245	-	0.0259	-	0.1339	-	0.0260	-	0.4517	-
0.0833	0.0552	2.0053	0.0172	1.0049	0.0596	1.9968	0.0172	1.0069	0.2967	1.0364
0.0625	0.0309	2.0092	0.0129	1.0063	0.0334	2.0039	0.0129	1.0074	0.2212	1.0195
0.0500	0.0197	2.0098	0.0103	1.0060	0.0214	2.0060	0.0103	1.0068	0.1765	1.0120
0.0416	0.0137	2.0094	0.0086	1.0055	0.0148	2.0065	0.0086	1.0060	0.1469	1.0081

Table 2.4: The computed errors for state, costate and control variables of semi-linear elliptic optimal control problem using DFV scheme with interpolated coefficients on a sequence of uniformly refined partition of $\Omega = (0, 1)^2$.

Newton method is employed to solve the resulting nonlinear system of equations arising from the discrete formulation. We compute the state and costate errors in L^2

and mesh dependent norm $|||\cdot|||_h$ and the control error in L^2-norm on a family of nested primal and dual triangulations of Ω. Here also we will use the similar notations as defined in (2.54) and (2.55) for control, state and costate errors in suitable norms and their respective convergence rates.

The interpolated coefficient DFV approximation errors $e_0(u), e_0(y), e_0(p), e_1(y)$ and $e_1(p)$ with three different control discretization techniques: variational discretization, piecewise linear and constant discretization are listed in Table 2.4. The expected approximation behaviour is observed.

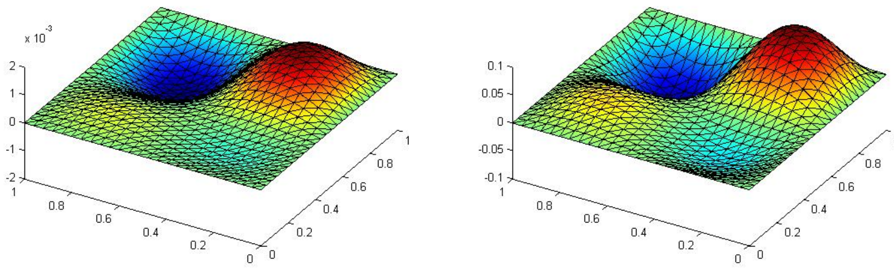
Application to the model problem

Example 2.8.2. The heat conduction optimal control problem (2.56)-(2.57) can be interpreted as an analogue of hyperthermia treatment for cancer. In medical science, hyperthermia (also called thermotherapy) is a type of cancer treatment in which body tissue is exposed to high temperatures that can kill or damage cancer cells. The domain Ω represents the cancerous tissue of the body. The control variable u corresponds to the heat source which distributed over the whole Ω. The ultimate goal is to destroy the cancer cells by driving the temperature distribution y as close as possible to the the desired temperature distribution y_d. We report on numerical tests for the optimal control problem (2.56)-(2.57) with $\Omega = (0,1) \times (0,1)$ and the given data

$$f = 0,\ y_d = sin(2\pi x_1)sin(2\pi x_2)exp(2x_1)/6,\ u_a = -2, u_b = 2,\ \lambda = 0.1, \varphi(y) = y^3.$$

The domain is discretized into 1250 primal triangulations with mesh size $h = 0.04$. Figure 2.9 plots the optimal control and associated optimal state obtained with interpolated coefficient DFV scheme with $\alpha = 10$ and piecewise linear control discretization. The iteration is stopped if the relative difference of two consecutive iterates is smaller than $1.e-6$. The method with this stopping criterion converges in five iterations. In addition, we observe that large number of iterations are required to reach the termination criteria for smaller values of regularization parameter (see Table 2.5)

Figure 2.9: The plots of the optimal control and state.



(a) The optimal state. (b) The optimal control.

λ	1	0.5	0.1	0.05	0.01
Iteration	4	4	5	7	11

Table 2.5: The iteration count for different values of control cost λ for the DFV approximations of semilinear elliptic optimal control problem with piecewise linear discretization of control.

CHAPTER 3

Semilinear parabolic optimal control problem

In this Chapter we discuss, in analogy to the elliptic case, the DFV approximations for the distributed optimal control problems governed by a class of semilinear parabolic partial differential equations with control constraints. For the spatial discretization of the state and costate variables, piecewise linear elements are used and an implicit finite difference scheme is used for time derivatives; whereas, for the approximation of the control variable, three different strategies are used: variational discretization, piecewise constant and piecewise linear discretization. Moreover, as our resulting DFV scheme leads to a non-symmetric discrete formulation, we have opted for *optimize-then-discretize* technique. *A priori* error estimates (for these three approaches) in suitable L^2-norm are derived for state, co-state and control variables. Numerical experiments are presented in order to assure the accuracy and rate of the convergence of the proposed scheme.

3.1 Introduction

The optimization of semilinear heat equations represent mathematical model for many physical applications, e.g. laser hardening, welding of steel, laser thermotherapy (used for cancer treatment) etc. In particular, the semilinear parabolic optimal control problems are used in describing a controlled non-stationary heat transfer process for optimal cooling of steel profiles. Finite element approximations for parabolic optimal control problems have been discussed in [61, 62, 76, 80] and references cited in these articles. Keeping in mind the applications of parabolic optimal control problems, we are interested in finding the numerical solution of the following optimization problem governed by semilinear heat equation with optimal control u and the associated optimal state y satisfying

$$\min_{u\in U_{ad}} J(y,u) := \frac{1}{2}\int_0^T \left(\|y(t,x) - y_d(t,x)\|_{0,\Omega}^2 + \lambda\,\|u(t,x)\|_{0,\Omega}^2\right) \mathrm{dt}, \tag{3.1}$$

subject to

$$\left.\begin{aligned} \partial_t y - \nabla \cdot \mathcal{A}\nabla y + \varphi(y) = \mathcal{B}u + f, \text{ in } (0,T) \times \Omega, \\ y(t,x) = 0, \text{ on } \quad (0,T) \times \partial\Omega, \\ y(0,x) = y_0(x), \qquad x \in \Omega, \end{aligned}\right\} \tag{3.2}$$

where, $T > 0$ is a given final time which defines the time interval $I := (0,T)$. As before, $\lambda > 0$ is the regularization parameter, $\mathcal{B}$ is a bounded linear operator and $\mathcal{A}$ is a real valued, symmetric and uniformly positive definite matrix defined in (2.3). For clarity, we usually suppress as in (3.2) the variables t and x in the functions y, u and the given data. Here and throughout this thesis, $\partial_t y$ denotes the partial derivative of y with respect to t. As the sets of feasible controls, we define

$$U_{ad} := \{u \in \mathcal{U} := L^\infty(L^\infty) : \ u_a \leq u \leq u_b, \ \ a.e. \, (t,x) \in (0,T) \times \Omega\}, \tag{3.3}$$

for control bounds $u_a, u_b \in \mathbb{R}$ with $u_a < u_b$.

In addition for our analysis, we require the following assumptions on the given data: we assume that the desired state $y_d(t,\cdot)$ and the source term $f \in L^\infty(\Omega)$ or $H^1(\Omega) \cap L^\infty(\Omega)$ for $0 < t \leq T$. As in [16, 67], we make the following assumptions for the nonlinear term φ. The function $\varphi(t,x,y) : I \times \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is of class C^2 and its first derivative φ' is nonnegative. For $y = 0$, φ and its derivatives upto second order are bounded by a positive constant. Moreover, on bounded sets, they are uniformly Lipschitz continuous with respect to y.

It is easy that for a fixed control $u \in L^\infty(\Omega)$, the state equation (3.2) exhibits a unique solution. By introducing the control-to-state mapping $\mathcal{S}$ (see [67, 79]) as $\mathcal{S}(u) = y$, the control problem (3.1)-(3.2) can be reduced to:

$$\min_{u \in U_{ad}} j(u) := \min_{u \in U_{ad}} J(\mathcal{S}(u), u). \tag{3.4}$$

By standard arguments the existence of at least one optimal control $u \in U_{ad}$ with associated optimal state $y = \mathcal{S}(u)$ for the optimal control problem (3.4) has been demonstrated in [67]. Due to non-convexity of reduced functional j, problem (3.4) may exhibit multiple solutions. Therefore, we consider the analysis for a fixed local reference solution defined below in sense of $L^2(L^2)$.

Definition 3.1.1. *A control $u \in U_{ad}$ is said to be the local solution of* (3.4) *in the sense of $L^2(L^2)$, if there exists a constant $\varepsilon > 0$ such that*

$$j(u) \le j(\tilde{u}), \quad \forall \tilde{u} \in U_{ad} \text{ with } \|\tilde{u} - u\|_{L^2(L^2)} \le \varepsilon.$$

The reduced objective functional j is of class C^2 (see [67]). The following Lemma formulates the first order necessary optimality conditions.

Lemma 3.1.2. *Every locally optimal control $u \in U_{ad}$ for the problem* (3.4) *in the sense of $L^2(L^2)$ satisfies the following variational inequality*

$$j'(u)(v - u) \ge 0, \quad \forall v \in U_{ad}. \tag{3.5}$$

The proof can be found in [79, Lemma 5.1]. The inequality (3.5) can also be rewritten in the form:

$$\int_0^T (\lambda u + \mathcal{B}^* p, \tilde{u} - u) \ge 0, \quad \forall \tilde{u} \in U_{ad}, \tag{3.6}$$

where p is the costate associated with u and solves the equation

$$\left.\begin{aligned} -\partial_t p - \nabla \cdot \mathcal{A} \nabla p + \varphi'(y) p = y - y_d, \quad & in \quad I \times \Omega, \\ p = 0, \quad & in \quad I \times \partial\Omega, \\ p(T, x) = 0, \quad & x \in \Omega. \end{aligned}\right\}$$

If we use the following pointwise projection on the admissible set U_{ad},

$$P_{[u_a,u_b]} : L^2(L^2) \longrightarrow U_{ad}, \quad P_{[u_a,u_b]}(z(t,x)) = \max(u_a, \min(u_b, z(t,x))),$$

then the optimality condition (3.6) can be simplified further as:

$$u(t,x) = P_{[u_a,u_b]} \left(\frac{-1}{\lambda} \mathcal{B}^* p(t,x) \right).$$

From the definition of the projection operator $P_{[u_a,u_b]}$, it is clear that this operator satisfies the following regularity properties

$$\left\|\nabla(P_{[u_a,u_b]}(v))(t)\right\|_{L^\infty(\Omega)} \leq \|\nabla v(t)\|_{L^\infty(\Omega)}, \quad \forall v \in L^2(W^{1,\infty}), \tag{3.7}$$

for almost all $t \in I$. For our forthcoming analysis, we formulate the following second order sufficient optimality condition.

Assumption 3.1.3. Let $u \in U_{ad}$ satisfies the first order necessary optimality conditions (3.5). Then we assume that there exists a positive constant C such that

$$j''(u)(\tilde{u}, \tilde{u}) \geq C \|\tilde{u}\|^2_{L^2(L^2)}, \quad \forall \tilde{u} \in \mathcal{U}, \tag{3.8}$$

which we refer as second order sufficient optimality condition (see [67]).

This Chapter is organized in the following manner. Section 3.1 is introductory and deals with statement of the governing problems and the standard optimality conditions. In Section 3.2, we apply DFV methods for the spatial discretization of the proposed optimal control problem (3.1)-(3.2) with three different discretization approaches for control variable. This Section also recalls some primary auxiliary results required for subsequent Sections. In Section 3.3, we derive *a priori* error estimates for semi-discrete scheme in suitable norms for state, costate and control variables. In Section 3.4, we analyze the error estimates for the fully discrete DFV approximation of the the control problem (3.1)-(3.2) Finally, in Section 3.5, we present some numerical experiments to justify the theoretical convergence rates and to illustrate the performance of the method.

3.2 Finite dimensional formulation

3.2.1 Semi-discrete scheme

For the spatial discretization of the control problem (3.1)-(3.2), we have used linear DFV methods. However, as mentioned earlier, for control discretization three different approaches: variational discretization, piecewise linear and piecewise constant discretization are used. As presented in Section 2.2.1 of Chapter 2, the DFV discretization in space consists of primal and dual triangulations of $\bar{\Omega}$, denoted by $\mathcal{T}_h$ and $\mathcal{T}_h^*$, respectively. We recall that the finite dimensional trial and test spaces associated with $\mathcal{T}_h$ and

$\mathcal{T}_h^*$, respectively are defined as follows:

$$V_h = \{v_h \in L^2(\Omega) : v_h|_K \in \mathcal{P}_1(K) \quad \forall K \in \mathcal{T}_h\},$$
$$V_h^* = \{v_h \in L^2(\Omega) : v_h|_{K^*} \in \mathcal{P}_0(K^*) \quad \forall K^* \in \mathcal{T}_h^*\},$$

with $\mathcal{P}_r(K)$ or $\mathcal{P}_r(K^*)$ denoting the space of all polynomials of degree less than or equal to r defined on K or K^*, respectively. Let U_h be a finite dimensional subspace of $L^2(\Omega)$, then the discrete admissible space for control is $U_{h,ad} = U_h \cap U_{ad}$ For discretization of control variable, here also, we consider the following three approaches:

1. **Variational approach.** In this approach, control variables are not discretized explicitly and the discrete admissible space $U_{h,ad}$ coincides with the space U_{ad}.

2. **Piecewise constant discretization**. Another approach for the discretization of the control variable is to use elementwise constant functions. In this case, the discrete control space is defined as

$$U_h = \{u_h(t,\cdot) \in L^2(\Omega) : u_h(t,\cdot)|_K \in \mathcal{P}_0(K), \quad \forall K \in \mathcal{T}_h, t \in I\}.$$

3. **Piecewise linear discretization.** Other natural way for seeking approximation of the control variable is to use piecewise linear functions on each element which is defined as

$$U_h = \{u_h(t,\cdot) \in L^2(\Omega) : u_h(t,\cdot)|_K \in \mathcal{P}_1(K), \quad \forall K \in \mathcal{T}_h, t \in I\}.$$

Applying the DFV scheme directly for the spatial discretization of the optimal control problem (3.1)-(3.2), we end up with the following semi-discrete formulation: Find

$(y_h(t,\cdot), p_h(t,\cdot), u_h(t,\cdot)) \in V_h \times V_h \times U_{h,ad}$ for $0 < t < T$ such that

$$(\partial_t y_h, \gamma v_h) + A_h(y_h, v_h) + (\varphi(y_h), \gamma v_h) = (\mathcal{B}u_h + f, \gamma v_h), \; \forall v_h \in V_h, \tag{3.9}$$
$$y_h(0,x) = y_{0,h}, \quad x \in \Omega,$$
$$-(\partial_t p_h, \gamma q_h) + A_h(p_h, q_h) + (\varphi'(y_h)p_h, \gamma q_h) = (y_h - y_d, \gamma q_h), \; \forall q_h \in V_h, \tag{3.10}$$
$$p_h(T,x) = 0, \quad x \in \Omega,$$
$$\int_0^T (\lambda u_h + \mathcal{B}^* p_h, \tilde{u}_h - u_h) \geq 0, \quad \forall \tilde{u}_h \in U_{h,ad}, \tag{3.11}$$

where the bilinear form $A_h(\cdot,\cdot)$ is defined in (2.15).

3.2.2 Fully-discrete scheme

For discretization of time derivative, backward Euler scheme is used, and for this we proceed as follows: let $0 = t_0 < t_1 < ... < t_M = T$ be a partition of time interval $[0,T]$ into subintervals $I_m = (t_{m-1}, t_m]$ with length $k_m = t_m - t_{m-1}$ for $m = 1, 2, ...M$ and $k = \max_{1 \leq m \leq M} k_m$. Now we use the backward Euler scheme which is defined as follows:

$$\partial_t v^m := \frac{(v^m - v^{m-1})}{k_m},$$

where $v^m = v(t_m, x)$.

In the light of the above mentioned discretization approaches for spatial and time domain, the backward Euler fully-discrete piecewise linear DFV formulation of the semilinear parabolic optimal control problem (3.1)-(3.2) is read as follows (see also [57, 60]): find $(y_h^m, p_h^{m-1}, u_h^m) \in V_h \times V_h \times U_{h,ad}$ such that $\forall \, v_h, q_h \in V_h$

$$(\partial_t y_h^m, \gamma v_h) + A_h(y_h^m, v_h) + (\varphi(y_h^m), \gamma v_h) = (\mathcal{B}u_h^m + f^m, \gamma v_h), \tag{3.12}$$
$$m = 1, ..., M; \qquad y_h^0(x) = y_{0,h}, \quad x \in \Omega,$$
$$-(\partial_t p_h^m, \gamma q_h) + A_h(p_h^{m-1}, q_h) + (\varphi'(y_h^m)p_h^{m-1}, \gamma q_h) = (y_h^m - y_d^m, \gamma q_h), \tag{3.13}$$
$$m = M, ..., 1; \qquad p_h^M(x) = 0, \quad x \in \Omega,$$
$$(\lambda u_h^m + \mathcal{B}^* p_h^{m-1}, \tilde{u}_h - u_h^m) \geq 0, \quad \forall \tilde{u}_h \in U_{h,ad}, \; m = 1, ..., M. \tag{3.14}$$

3.3 Error estimates for semi-discrete scheme

In this Section, we derive error estimates for a fixed local (in the sense of $L^2(L^2)$) reference solution of the problem (3.4) which also satisfy first and second order optimality conditions. Since the control and state variables u and y appears in the state and costate equations, respectively, the error estimates for state and costate variables depend on the control variable and state variables, respectively.

For deriving these estimates we proceed in the following way. For a given arbitrary $\tilde{u} \in L^2(L^2)$ and $\tilde{y} = y(\tilde{u}) \in L^2(H_0^1)$, let $y_h(\tilde{u})(t,\cdot)$ and $p_h(\tilde{y})(t,\cdot)$ be the solutions of the following equations for $v_h, q_h \in V_h$ and $0 \leq t \leq T$.

$$(\partial_t y_h(\tilde{u}), \gamma v_h) + A_h(y_h(\tilde{u}), v_h) + (\varphi(y_h(\tilde{u})), \gamma v_h) = (\mathcal{B}\tilde{u} + f, \gamma v_h) \tag{3.15}$$
$$y_h(\tilde{u})(0,x) = y_{0,h}, \quad x \in \Omega,$$

and

$$-(\partial_t p_h(\tilde{y}), \gamma q_h) + A_h(p_h(\tilde{y}), q_h) + (\varphi'(y)p_h(\tilde{y}), \gamma q_h) = (\tilde{y} - y_d, \gamma q_h), \tag{3.16}$$
$$p_h(\tilde{y})(T,x) = 0, \quad x \in \Omega,$$

respectively.

We will frequently use the following notations for our analysis. For simplicity, we denote by $L^s(L^p)$, $1 \leq s,p < \infty$, the Banach space of all functions $\phi(t) : [0,T] \longrightarrow L^p(\Omega)$, such that $\|\phi(t)\|_{L^p(\Omega)} \in L^s(0,T)$ and associated with the following norm

$$\|\phi\|_{L^s(L^p)} := \left(\int_0^T \|\phi(t)\|^s_{L^p(\Omega)} \right)^{1/s} \qquad s \in [1,\infty).$$

and the standard modification for $s = \infty$. Let $L^s(V(h))$, $1 \leq s < \infty$ denote the Banach space of all functions $\psi(t) : [0,T] \longrightarrow V(h)$, where $V(h) = V_h + H^2(\Omega) \cap H_0^1(\Omega)$ such that $|||\psi(t)|||_h \in L^s(0,T)$ with the following norm

$$\|\psi\|_{L^s(V(h))} := \left(\int_0^T |||\psi(t)|||^s_h \right)^{1/s} \qquad s \in [1,\infty).$$

and the standard modification for $s = \infty$.

In order to avoid confusion, in what follows we will use the following notations: $y_h = y_h(u_h)$, $p_h = p_h(y_h)$ and $p_h(u) = p_h(y_h(u))$. Now using similar arguments as in the proof of [60, Lemma 4.1], we now prove the following Lemma for $\tilde{u} = u$ and $\tilde{y} = y(u)$.

Lemma 3.3.1. *There exists a positive constant C independent of h such that*

$$\|y_h(u) - y_h\|_{L^\infty(V(h))} \leq C \|u - u_h\|_{L^2(L^2)}, \quad \|p_h(y) - p_h\|_{L^\infty(V(h))} \leq C \|y - y_h\|_{L^2(L^2)}.$$

Proof. Subtracting equations (3.9) from (3.15), we obtain for all $v_h \in V_h$

$$(\partial_t(y_h(u) - y_h), \gamma v_h) + A_h(y_h(u) - y_h, v_h) + (\varphi(y_h(u)) - \varphi(y_h), \gamma v_h) = (\mathcal{B}(u - u_h), \gamma v_h).$$

Let us denote $y_h(u) - y_h = \eta$ then by choosing $v_h = \partial_t \eta$, the above equation can be rewritten as follows:

$$(\partial_t \eta, \gamma \partial_t \eta) + A_h(\eta, \partial_t \eta) = (\mathcal{B}(u - u_h), \gamma \partial_t \eta) + (\varphi(y_h) - \varphi(y_h(u)), \gamma \partial_t \eta).$$

Using the definitions of the norm $|||\cdot|||_0$ and $\epsilon_a(\cdot, \cdot)$, we arrive at

$$\begin{aligned} |||\partial_t \eta|||_0^2 + \frac{1}{2}\frac{d}{dt} a_h(\eta, \eta) =& (\mathcal{B}(u - u_h), \gamma \partial_t \eta) + \epsilon_a(\eta, \partial_t \eta) \\ &+ (\varphi(y_h) - \varphi(y_h(u)), \gamma(\partial_t \eta)). \end{aligned}$$

Integrating from 0 to t and noting that $\eta(0, x) = 0$

$$\begin{aligned} 2\int_0^t |||\partial_t \eta|||_0^2 \, d\tau + a_h(\eta, \eta) =& 2\int_0^t (\mathcal{B}(u - u_h), \gamma \partial_t \eta) d\tau + 2\int_0^t \epsilon_a(\eta, \partial_t \eta) d\tau \\ &+ 2\int_0^t (\varphi(y_h) - \varphi(y_h(u)), \gamma(\partial_t \eta)) d\tau. \end{aligned}$$

The coercive property of $a_h(\cdot, \cdot)$ implies

$$\begin{aligned} 2\int_0^t |||\partial_t \eta|||_0^2 \, d\tau + C \, |||\eta|||_h^2 \leq& 2\int_0^t (\mathcal{B}(u - u_h), \gamma \partial_t \eta) d\tau + 2\int_0^t \epsilon_a(\eta, \partial_t \eta) d\tau \\ &+ 2\int_0^t (\varphi(y_h) - \varphi(y_h(u)), \gamma(\partial_t \eta)) d\tau. \end{aligned} \tag{3.17}$$

An application of (2.19) and inverse inequality together with Young's inequality, provide us

$$\epsilon_a(\eta, \partial_t \eta) \leq Ch \left\|\!\left|\eta\right|\!\right\|_h \left\|\!\left|\partial_t \eta\right|\!\right\|_h \leq C \left\|\!\left|\eta\right|\!\right\|_h \left\|\partial_t \eta\right\|_{0,\Omega} \leq C(\epsilon) \left\|\!\left|\eta\right|\!\right\|_h^2 + \epsilon \left\|\partial_t \eta\right\|_{0,\Omega}^2. \quad (3.18)$$

The following inequality follows by Cauchy-Schwarz inequality and result (2.10) of Lemma 2.2.1

$$\begin{aligned}(\mathcal{B}(u-u_h), \gamma\partial_t \eta) &\leq C \left\|u-u_h\right\|_{0,\Omega} \left\|\gamma \partial_t \eta\right\|_{0,\Omega} \leq C \left\|u-u_h\right\|_{0,\Omega} \left\|\partial_t \eta\right\|_{0,\Omega} \\ &\leq C(\epsilon) \left\|u-u_h\right\|_{0,\Omega}^2 + \epsilon \left\|\partial_t \eta\right\|_{0,\Omega}^2. \quad (3.19)\end{aligned}$$

The Lipschitz continuity of $\varphi(\cdot)$ together with (2.17) implies that

$$\begin{aligned}(\varphi(y_h) - \varphi(y_h(u)), \gamma \partial_t \eta) &\leq \left\|\eta\right\|_{0,\Omega} \left\|\gamma \partial_t \eta\right\|_{0,\Omega} \leq \left\|\!\left|\eta\right|\!\right\|_h \left\|\partial_t \eta\right\|_{0,\Omega} \\ &\leq C(\epsilon) \left\|\!\left|\eta\right|\!\right\|_h^2 + \epsilon \left\|\partial_t \eta\right\|_{0,\Omega}^2. \quad (3.20)\end{aligned}$$

Collecting the bounds obtained in (3.18), (3.19), (3.20) and using equivalence of $\left\|\!\left|\cdot\right|\!\right\|_0$ and $\left\|\cdot\right\|_{0,\Omega}$ with appropriate ϵ in relation (3.17), enable us to write the following

$$\int_0^t \left\|\partial_t \eta\right\|_{0,\Omega}^2 d\tau + \left\|\!\left|\eta\right|\!\right\|_h^2 \leq C \int_0^t \left\|\!\left|\eta\right|\!\right\|_h^2 d\tau + C \int_0^T \left\|u-u_h\right\|_{0,\Omega}^2 d\tau. \quad (3.21)$$

Dropping the first term and applying Gronwall's inequality implies that $\left\|\eta\right\|_{L^\infty(V(h))} \leq C \left\|u-u_h\right\|_{L^2(L^2)}$, which completes the proof of first required result. For estimating the second result, we proceed similarly by subtracting (3.10) from (3.16), writing $p_h(y) - p_h = \mu$ and choosing $q_h = \partial_t \mu$, we infer that

$$\left\|\!\left|\partial_t \mu\right|\!\right\|_0^2 - \frac{1}{2}\frac{d}{dt} a_h(\mu, \mu) = (y_h - y, \gamma \partial_t \mu) - \epsilon_a(\mu, \partial_t \mu) - (\phi'(y) p_h(y) - \phi'(y_h) p_h, \gamma \partial_t \mu).$$

Integrating both sides from t to T, and noting $\mu(T, x) = 0$, we have

$$\begin{aligned}2\int_t^T \left\|\!\left|\partial_t \mu\right|\!\right\|_0^2 d\tau + a_h(\mu, \mu) =& 2\int_t^T (y_h - y, \gamma \partial_t \mu) d\tau + 2\int_t^T \epsilon_a(\mu, \partial_t \mu) d\tau \\ &- 2\int_t^T (\phi'(y) p_h(y) - \phi'(y_h) p_h, \gamma \partial_t \mu) d\tau.\end{aligned}$$

The coercivity of $a_h(\cdot,\cdot)$ implies that

$$\int_t^T |||\partial_t\mu|||_0^2\, d\tau + |||\mu|||_h^2 \leq C\int_t^T (y_h - y, \gamma\partial_t\mu) d\tau + C\int_t^T \epsilon_a(\mu, \partial_t\mu) d\tau$$
$$+C\int_t^T (\phi'(y)p_h(y) - \phi'(y_h)p_h, \gamma\partial_t\mu) d\tau. \qquad (3.22)$$

It follows from the assumption $\phi'(\cdot) \geq 0$ and Lipschitz continuity that

$$(\phi'(y)p_h(y) - \phi'(y_h)p_h, \gamma\partial_t\mu) \leq C(\mu, \gamma\partial_t\mu) \leq C\|\mu\|\,\|\gamma\partial_t\mu\|_{0,\Omega}$$
$$\leq C(\epsilon)\,|||\mu|||_h^2 + \epsilon\,\|\partial_t\mu\|_{0,\Omega}^2. \qquad (3.23)$$

Also, using the arguments used in derivation of inequalities (3.18) and (3.20), we have the following bounds

$$\epsilon_a(\mu, \partial_t\mu) \leq Ch\,|||\mu|||_h\,|||\partial_t\mu|||_h \leq C(\epsilon)\,|||\mu|||_h^2 + \epsilon\,\|\partial_t\mu\|_{0,\Omega}^2, \qquad (3.24)$$
$$(y_h - y, \gamma\partial_t\mu) \leq C\|y - y_h\|_{0,\Omega}\|\gamma\partial_t\mu\|_{0,\Omega} \leq C(\epsilon)\|y - y_h\|_{0,\Omega}^2 + \epsilon\,\|\partial_t\mu\|_{0,\Omega}^2. \qquad (3.25)$$

Using inequalities (3.23), (3.24), (3.25) and equivalence of $|||\cdot|||_0$ and $\|\cdot\|_{0,\Omega}$ alongwith appropriate ϵ in relation (3.22), we can find that

$$|||\mu|||_h^2 \leq C\int_t^T |||\mu|||_h^2\, d\tau + C\int_0^T \|y - y_h\|_{0,\Omega}^2\, d\tau.$$

Now an application to Gronwall's inequality implies the required estimate, i.e., $\|\mu\|_{L^\infty(V(h))} \leq C\|y - y_h\|_{L^2(L^2)}$. □

Lemma 3.3.2. *There exists a positive constant C independent of h such that the following relation holds:*

$$\|\partial_t(y_h(u) - y_h)\|_{L^\infty(L^2)} \leq C\|\partial_t(u - u_h)\|_{L^2(L^2)}.$$

Proof. Differentiating (3.2) with respect to t and multiplying by γv_h, we can obtain the following relation by employing discrete state equation for y_h and $y_h(u)$

$$(\partial_{tt} y_h(u) - \partial_{tt} y_h, \gamma v_h) + A_h(\partial_t(y_h(u) - y_h), v_h) + (\phi'(y_h(u))\partial_t y_h(u) - \phi'(y_h)\partial_t y_h, \gamma v_h) \\ = (\mathcal{B}\partial_t(u - u_h), \gamma v_h).$$

Denoting $y_h(u) - y_h = \mu$ and choosing $v_h = \partial_t \mu$ in the above equation, we have

$$\frac{1}{2}\frac{d}{dt}(\partial_t \mu, \gamma \partial_t \mu) + A_h(\partial_t \mu, \partial_t \mu) = (\mathcal{B}\partial_t(u - u_h), \partial_t \mu) \\ - (\phi'(y_h(u))\partial_t y_h(u) - \phi'(y_h)\partial_t y_h, \gamma \partial_t \mu).$$

Integrating from 0 to t, using the coercivity of $A_h(\cdot,\cdot)$ and monotonicity of nonlinear term, we can obtain

$$\|\partial_t \mu\|_{0,\Omega}^2 + C\int_0^t \|\mu\|_h^2 \, d\tau \le C\int_0^T \|\partial_t(u - u_h)\|_{0,\Omega}^2 \, d\tau + C\int_0^t \|\partial_t \mu\|_{0,\Omega}^2 \, d\tau.$$

Dropping the second term and using Gronwall's Lemma we get $\|\partial_t(y_h(u) - y_h)\|_{L^\infty(L^2)} \le C\,\|\partial_t(u - u_h)\|_{L^2(L^2)}$. □

We define the elliptic projection $R_h : H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow V_h$ by

$$A_h(R_h u, v_h) := A_h(u, v_h) \quad \forall v_h \in V_h, \tag{3.26}$$

and in what follows, we choose $y_{0,h} = R_h y_0(x)$ for $x \in \Omega$. Now, for a given u, the following estimates can be derived by using the elliptic projection defined in (3.26) and appealing to duality arguments used for the standard DFV analysis for parabolic problems. Therefore, we refrain ourself for providing this proof and we refer to [9], also see [50] and [12].

Lemma 3.3.3. *For any $\tilde{u} \in L^2(L^2)$, $\tilde{y} = y(\tilde{u}) \in L^2(H^2)$, there exists a positive constant C independent of h such that*

$$\|y(\tilde{u}) - y_h(\tilde{u})\|_{L^2(V(h))} + \|p(\tilde{y}) - p_h(\tilde{y})\|_{L^2(V(h))} \le Ch.$$

Lemma 3.3.4. *For any $\tilde{u} \in L^2(L^2)$ and $\tilde{y} = y(\tilde{u}) \in L^2(H^3)$, there exists a positive* *constant C independent of h such that*

$$\|y(\tilde{u}) - y_h(\tilde{u})\|_{L^2(L^2)} + \|p(\tilde{y}) - p_h(\tilde{y})\|_{L^2(L^2)} + \|p(\tilde{u}) - p_h(\tilde{u})\|_{L^2(L^2)} \leq Ch^2,$$

and in particular, for $\tilde{u} = u_h$, we will have

$$\|p(u_h) - p_h(u_h)\|_{L^2(L^2)} \leq Ch^2. \tag{3.27}$$

3.3.1 Error estimates for control

In this Section, we discuss convergence analysis for control variable with three discretization approaches.

With variational discretization:

We are now in a position to prove the following result of this Section.

Theorem 3.3.5. *Let u be a fixed local optimal control of problem (3.4) and u_h be the solution of the semidiscrete optimal control problem (3.9)-(3.10) with variational discretization approach, then the following error estimate holds.*

$$\|u - u_h\|_{L^2(L^2)} \leq Ch^2.$$

Proof. For variational discretization approach the discrete and continuous variational inequalities satisfies the relation

$$\int_0^T (\lambda u_h + \mathcal{B}^* p_h, u - u_h) d\tau \geq 0 \geq \int_0^T (\lambda u + \mathcal{B}^* p, u - u_h) d\tau.$$

The condition (3.8) for $u - u_h \in U_{ad} \subset L^2(L^2)$, implies that

$$\begin{aligned}\|u - u_h\|^2_{L^2(L^2)} &\leq \int_0^T \left[(\lambda u_h + \mathcal{B}^* p_h, u - u_h) - (\lambda u_h + \mathcal{B}^* p(u_h), u - u_h)\right] d\tau \\ &\leq C \|p(u_h) - p_h\|_{L^2(L^2)} \|u - u_h\|_{L^2(L^2)}.\end{aligned}$$

Using (3.27) in the above relation yields the required result, i.e.,

$$\|u - u_h\|_{L^2(L^2)} \leq Ch^2. \tag{3.28}$$

□

With piecewise constant discretization:

Now, we will derive the error estimates for $\|u - u_h\|_{L^2(L^2)}$ when the control variable is discretized by piecewise constants. For the accomplishment of the main result of this Section, we follow the similar idea used for elliptic problem by introducing the L^2-projection operator Π_0 onto discrete control space U_h which satisfy $\Pi_0 U_{ad} \subset U_{h,ad}$ and the approximation property (2.30).

Theorem 3.3.6. *Let u be a fixed local optimal control of problem (3.4) and u_h be the solution of the semi-discrete problem (3.9)-(3.10) with piecewise constant discretization of controls, then we have the following discretization error estimate*

$$\|u - u_h\|_{L^2(L^2)} \leq Ch.$$

Proof. Since $\Pi_0 U_{ad} \subset U_{h,ad}$, we easily see that the following holds with the help of continuous and discrete optimality conditions

$$\int_0^T (\lambda u_h + \mathcal{B}^* p_h, \Pi_0 u - u_h) d\tau \geq 0 \geq \int_0^T (\lambda u + \mathcal{B}^* p, u - u_h) d\tau. \tag{3.29}$$

Applying condition (3.8) for $u - u_h \in U_{ad} \subset L^2(L^2)$ and using inequalities in (3.29), we can obtain

$$\begin{aligned} C \|u - u_h\|^2_{L^2(I;L^2(\Omega))} &\leq \int_0^T \left((\lambda u_h + \mathcal{B}^* p_h, \Pi_0 u - u_h) - (\lambda u_h + \mathcal{B}^* p(u_h), u - u_h) \right) d\tau \\ &\leq \underbrace{\int_0^T (\mathcal{B}^* p_h - B^* p(u_h), u - u_h) d\tau}_{J_1} + \underbrace{\int_0^T (\lambda u_h + \mathcal{B}^* p_h, \Pi_0 u - u) d\tau}_{J_2}. \end{aligned} \tag{3.30}$$

The property (3.27) and continuity of operator $\mathcal{B}$, yields

$$J_1 \leq \|p(u_h) - p_h\|_{L^2(L^2)} \|u - u_h\|_{L^2(L^2)} \leq Ch^2 \|u - u_h\|_{L^2(L^2)} .$$

To achieve the desried bound for J_2 we use the property of the projection Π_0 to rewrite it as:

$$J_2 = \int_0^T (\mathcal{B}^* p_h - \Pi_0(\mathcal{B}^* p_h), \Pi_0 u - u) d\tau \leq \|\mathcal{B}^* p_h - \Pi_0(\mathcal{B}^* p_h)\|_{L^2(L^2)} \|\Pi_0 u - u\|_{L^2(L^2)}$$
$$\leq Ch^2 \|p_h\|_{L^2(V(h))} \|u\|_{L^2(H^1)} .$$

Now, we need to show that the p_h is uniformly bounded. Testing the discrete state equation (3.9) for $v_h = \partial_t y_h$, making use of coercivity of the bilinear form $A_h(\cdot, \cdot)$ with respect to the norm $|||\cdot|||_h$, estimate (2.19), properties of nonlinear term and Gronwall's inequality, we can achieve

$$\|y_h\|_{L^2(V(h))} \leq C \left(\|u_h\|_{L^2(L^2)} + \|f\|_{L^2(L^2)} \right) .$$

Similarly, from the discrete state equation, we can obtain

$$\|p_h\|_{L^2(V(h))} \leq C \left(\|y_h\|_{L^2(L^2)} + \|y_d\|_{L^2(L^2)} \right) .$$

The above two equations alongwith uniform boundedness of $U_{h,ad}$ implies p_h is uniformly bounded. Therefore, on substituting the bounds for J_1 and J_2 in (3.30) and applying Young's inequality, we complete the rest of the proof. □

With piecewise linear discretization:

We first lay out some assumptions on the structure of the active sets. Let $\tilde{u}_h(t, \cdot)$ be a function in discrete admissible set $U_{h,ad}$ defined on an arbitrary triangle $K \in \mathcal{T}_h$ for

$0 < t \leq T$ by

$$\tilde{u}_h(t,x) = \begin{cases} u_a & \text{if } \min_{x\in K} u(t,x) = u_a, \\ u_b & \text{if } \max_{x\in K} u(t,x) = u_b, \\ \tilde{I}_h u & \text{else.} \end{cases} \tag{3.31}$$

Here, $\tilde{I}_h u$ represents Lagrange interpolate of u. To avoid confusion, the mesh size h is chosen sufficiently small so that $\min_{x\in K} u(t,x) = u_a$ and $\max_{x\in K} u(t,x) = u_b$ cannot happen simultaneously in the same element K. Now, the elements $K \in \mathcal{T}_h$ are grouped into three sets $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2 \cup \mathcal{T}_h^3$ with $\mathcal{T}_h^i \cap \mathcal{T}_h^j = \emptyset$ for $i \neq j$ according to the value of $u(t,x)$ on K. These sets are defined as follows:

$$\begin{aligned} \mathcal{T}_h^1 &= \{K \in \mathcal{T}_h : u(t,x) = u_a \quad or \quad u(t,x) = u_b \quad \forall x \in K\}, \\ \mathcal{T}_h^2 &= \{K \in \mathcal{T}_h : u_a < u(t,x) < u_b \quad \forall x \in K\}, \\ \mathcal{T}_h^3 &= \mathcal{T}_h \setminus (\mathcal{T}_h^1 \cup \mathcal{T}_h^2). \end{aligned}$$

As seen before, with the help of definition (3.31) it follows that for any $\tilde{u}_h \in U_{h,ad}$ we have

$$\int_0^T (\lambda u + \mathcal{B}^* p, \tilde{u} - \tilde{u}_h) d\tau \geq 0, \quad \forall \tilde{u} \in U_{ad}. \tag{3.32}$$

For our subsequent analysis, we will exploit the following assumption

Assumption 3.3.7. There exists a positive constant C independent of h such that

$$\sum_{K\in\mathcal{T}_h^3} |K| \leq Ch.$$

Now we state the following assertion which will be used in deriving the error estimates for control with piecewise linear discretization. The proof can be found in [62].

Lemma 3.3.8. *Let u be a local control of the optimization problem* (3.4)*. Then, under* *the assumption* (3.3.7)*, the following estimate holds, provided* $p \in L^2(W^{1,\infty})$*:*

$$\int_0^T |(\lambda u + \mathcal{B}^* p, \tilde{u}_h - u)| d\tau \leq \frac{C}{\lambda} h^3 \|\nabla p\|^2_{L^2(L^\infty)}, \quad \forall \tilde{u}_h \in U_{h,ad}.$$

Following the same idea used in the establishment of [62, Lemma 5.7], we prove our main result.

Theorem 3.3.9. *Let u be a fixed local optimal control of problem* (3.4) *and u_h be the solution of the semi-discrete optimal control problem* (3.9)-(3.10) *with piecewise linear discretization of controls then the following estimate holds true*

$$\|u - u_h\|_{L^2(L^2)} \leq Ch^{3/2}.$$

Proof. From the continuous and discrete variational inequalities we have

$$\int_0^T (\lambda u_h + \mathcal{B}^* p_h, \tilde{u}_h - u_h) d\tau \geq 0 \geq \int_0^T (\lambda u + \mathcal{B}^* p, u - u_h) d\tau. \tag{3.33}$$

Using the second order sufficient condition (3.8) for $u - u_h \in U_{h,ad}$ and (3.33), we have

$$\begin{aligned}
C \|u - u_h\|^2_{L^2(L^2)} &\leq \int_0^T \left[(\lambda u + \mathcal{B}^* p, u - u_h) - (\lambda u_h + \mathcal{B}^* p(u_h), u - u_h)\right] d\tau \\
&\leq \int_0^T [(\lambda(u - u_h) + \mathcal{B}^*(p - p_h), u - \tilde{u}_h) + (\lambda u + \mathcal{B}^* p, \tilde{u}_h - u)) \\
&\quad + (p_h - p(u_h), \mathcal{B}(u - u_h))] d\tau \\
&\leq \left(\|u - u_h\|_{L^2(L^2)} + \|p - p_h\|_{L^2(L^2)}\right) \|u - \tilde{u}_h\|_{L^2(L^2)} \\
&\quad + Ch^2 \|u - u_h\|_{L^2(L^2)} + \int_0^T |(\lambda u + \mathcal{B}^* p, \tilde{u}_h - u)| d\tau.
\end{aligned} \tag{3.34}$$

As derived in the Theorem 2.3.9 of Chapter 2, by using the definition 3.31 and projection property 3.7 together with Assumption 3.3.7, the following estimate can be derived

$$\|u - \tilde{u}_h\|_{L^2(L^2)} \leq \frac{C}{\lambda}[h^2 \|\nabla^2 p\|_{L^2(L^2)} + h^{3/2} \|\nabla p\|_{L^2(L^\infty)}]. \tag{3.35}$$

Using triangle inequality along with property (2.17) and results of Lemma 3.3.1 and

Lemma 3.3.4, we find that

$$
\begin{aligned}
\|p-p_h\|_{L^2(L^2)} &\leq \|p-p_h(y)\|_{L^2(L^2)} + \|p_h(y)-p_h\|_{L^\infty(V(h))} \\
&\leq \|p-p_h(y)\|_{L^2(L^2)} + C(\|y-y_h(u)\|_{L^2(L^2)} \\
&+ \|y_h(u)-y_h\|_{L^\infty(V(h))}) \\
&\leq Ch^2 + C\,\|u-u_h\|_{L^2(L^2)}\,.
\end{aligned}
$$

Inserting the above relation and (3.35) in (3.34) after applying Young's inequality and using the results of Lemma 3.3.8, we can obtain the desired estimate $\|u-u_h\|_{L^2(L^2)} = \mathcal{O}(h^{\frac{3}{2}})$. □

3.3.2 Error estimates for state and costate

Under variational discretization of control:

Theorem 3.3.10. *Let u be a local optimal control of problem (3.4) with the associated state y and costate p, respectively, and let u_h, y_h and p_h be the solution of the semi-discrete optimal control problem (3.9)-(3.10) with variational approach, then the following discretization error estimates for state and costate variables are satisfied*

$$\|y-y_h\|_{L^2(L^2)} + \|p-p_h\|_{L^2(L^2)} \leq Ch^2.$$

Proof. First we decompose the error for state as $y-y_h = y-y_h(u)+y_h(u)-y_h$ and for costate as $p-p_h = p-p_h(y)+p_h(y)-p_h$. Now, the following estimates for state and costate errors directly follows with an application of triangle inequality together with the results of Lemmas 3.3.1, 3.3.4 and the estimate (3.28) for control error in $L^2(L^2)$-norm

$$\|y-y_h\|_{L^2(L^2)} + \|p-p_h\|_{L^2(L^2)} \leq Ch^2.$$

□

Under explicit discretization of control:

We would like to mention that for variational discretization we are able to derive optimal error estimates (for state and costate) with the help of Lemmas 3.3.1 and 3.3.4. But if we proceed in the similar way we end up with the convergence of $\mathcal{O}(h^{3/2})$ and $\mathcal{O}(h)$ for piecewise linear and constant discretization approaches, respectively. In order to achieve the desired optimal estimates for both piecewise linear and constant discretizations for control, we appeal to duality arguments in the following main Theorem of this Section. The similar idea also used in [67] and [61].

Theorem 3.3.11. *Let u be a local optimal control of problem (3.4) with the associated state y and costate p, respectively, and let u_h, y_h and p_h be the solution of the semi-discrete optimal control problem (3.9)-(3.10) under piecewise constant (or linear) control discretization, then the following discretization error estimates are satisfied*

$$\|y - y_h\|_{L^2(L^2)} + \|p - p_h\|_{L^2(L^2)} \leq Ch^2.$$

Proof. Splitting the error $y - y_h = y - y_h(u) + y_h(u) - y_h(\Pi_h u) + y_h(\Pi_h u) - y_h$ and applying triangle inequality we can write

$$\|y - y_h\|_{L^2(L^2)} \leq \|y - y_h(u)\|_{L^2(L^2)} + \|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)} + \|y_h(\Pi_h u) - y_h\|_{L^2(L^2)}. \tag{3.36}$$

Here, Π_h denotes the L^2 projection operator onto U_h. Now, let us assume that $\tilde{p}_h(t,\cdot) \in V_h$, $(0 < t \leq T)$ be the solution of auxiliary discrete dual equation

$$\begin{aligned} -(\xi, \partial_t \tilde{p}_h) + a_h(\tilde{p}_h, \xi) &= (\xi, y_h(u) - y_h(\Pi_h u)) - (\xi, \hat{\phi}\tilde{p}_h), \quad \forall \xi \in V_h \\ \tilde{p}_h(T, x) &= 0, \end{aligned} \tag{3.37}$$

with

$$\hat{\phi}(t,x) = \begin{cases} \frac{\varphi(y_h(u)) - \varphi(y_h(\Pi_h u))}{y_h(u) - y_h(\Pi_h u)}, & \text{if} \quad y_h(u) \neq y_h(\Pi_h u)) \\ 0, & \text{else.} \end{cases}$$

We note that $\left\|\hat{\phi}\right\|_{L^\infty(L^\infty)} \leq c$ for a $c > 0$ due to boundedness of U_{ad}. Choosing $\xi = \partial_t \tilde{p}_h$ in (3.37), using coercivity of $a_h(\cdot,\cdot)$ and applying Gronwall's Lemma, it is easy to check that the following result holds

$$\|\tilde{p}_h\|_{L^\infty(V(h))} \leq C \|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)}. \tag{3.38}$$

Testing (3.37) against $\xi = y_h(u) - y_h(\Pi_h u)$, we find that

$$\begin{aligned} &-(y_h(u) - y_h(\Pi_h u), \partial_t \tilde{p}_h) + a_h(\tilde{p}_h, y_h(u) - y_h(\Pi_h u)) \\ &\quad = \|y_h(u) - y_h(\Pi_h u))\|_{0,\Omega}^2 - (\varphi(y_h(u)) - \varphi(y_h(\Pi_h u)), \tilde{p}_h). \end{aligned} \tag{3.39}$$

Employing the discrete state equation for $y_h(u)$ and $y_h(\Pi_h u)$, we have

$$\begin{aligned} &(\partial_t(y_h(u) - y_h(\Pi_h u)), \gamma\tilde{p}_h) + A_h(y_h(u) - y_h(\Pi_h u), \tilde{p}_h) \\ &\quad = (\mathcal{B}(u - \Pi_h u), \gamma\tilde{p}_h) - (\varphi(y_h(u)) - \varphi(y_h(\Pi_h u)), \gamma\tilde{p}_h). \end{aligned} \tag{3.40}$$

Subtracting (3.40) from (3.39), integrating from 0 to T and rearranging the terms we can obtain

$$\begin{aligned} &\|y_h(u) - y_h(\Pi_h u))\|_{L^2(L^2)}^2 \\ &= \int_0^T (\mathcal{B}(u - \Pi_h u), \gamma\tilde{p}_h) d\tau + \int_0^T (\partial_t(y_h(u) - y_h(\Pi_h u)), \tilde{p}_h - \gamma\tilde{p}_h) d\tau \\ &+ \int_0^T \epsilon_a(y_h(u) - y_h(\Pi_h u), \tilde{p}_h) d\tau + \int_0^T (\varphi(y_h(u)) - \varphi(y_h(\Pi_h u)), \tilde{p}_h - \gamma\tilde{p}_h) d\tau \\ &=: S_1 + S_2 + S_3 + S_4. \end{aligned} \tag{3.41}$$

Using the property of projection Π_h, estimate (2.11) of γ and (3.38) we can get

$$\begin{aligned} S_1 &= \int_0^T \left[(\mathcal{B}(u - \Pi_h u), \gamma\tilde{p}_h - \tilde{p}_h) + (u - \Pi_h u, \mathcal{B}^*\tilde{p}_h - \Pi_h \mathcal{B}^*\tilde{p}_h) \right] d\tau \\ &\leq Ch \|u - \Pi_h u\|_{L^2(L^2)} \|\tilde{p}_h\|_{L^2(V(h))} \\ &\leq Ch \|u - \Pi_h u\|_{L^2(L^2)} \|\tilde{p}_h\|_{L^\infty(V(h))} \\ &\leq Ch \|u - \Pi_h u\|_{L^2(L^2)} \|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)}. \end{aligned}$$

Following similar steps as in the proof of Lemma 3.3.1 and Lemma 3.3.2, it is easy to establish

$$\|y_h(u) - y_h(\Pi_h u)\|_{L^\infty(V(h))} \le \|u - \Pi_h u\|_{L^2(L^2)}\,, \tag{3.42}$$

$$\|\partial_t(y_h(u) - y_h(\Pi_h u))\|_{L^\infty(L^2)} \le \|\partial_t(u - \Pi_h u)\|_{L^2(L^2)}\,. \tag{3.43}$$

Using the approximation property of γ as mentioned in Lemma 2.2.1, above result (3.43) and result (3.38) readily gives

$$\begin{aligned} S_2 &\le Ch\,\|\partial_t(y_h(u) - y_h(\Pi_h u))\|_{L^2(L^2))}\,\|\tilde{p}_h\|_{L^2(V(h))} \\ &\le Ch\,\|\partial_t(y_h(u) - y_h(\Pi_h u))\|_{L^\infty(L^2)}\,\|\tilde{p}_h\|_{L^\infty(V(h))} \\ &\le Ch\,\|\partial_t(u - \Pi_h u)\|_{L^2(L^2)}\,\|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)}\,. \end{aligned}$$

From the estimate of $\epsilon_a(\cdot,\cdot)$ in Lemma 2.2.2 and using (3.38), one can obtain

$$\begin{aligned} S_3 &\le Ch\,\|y_h(u) - y_h(\Pi_h u)\|_{L^2(V(h))}\,\|\tilde{p}_h\|_{L^2(V(h))} \\ &\le Ch\,\|y_h(u) - y_h(\Pi_h u)\|_{L^\infty(V(h))}\,\|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)} \\ &\le Ch\,\|u - \Pi_h u\|_{L^2(L^2)}\,\|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)}\,. \end{aligned}$$

To bound S_4 we use Lipschitz continuity of nonlinear term $\varphi(\cdot)$, estimate of γ and (3.38)

$$\begin{aligned} S_4 &\le Ch\,\|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)}\,\|\tilde{p}_h\|_{L^2(V(h))} \\ &\le Ch\,\|y_h(u) - y_h(\Pi_h u)\|_{L^\infty(V(h))}\,\|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)} \\ &\le Ch\,\|u - \Pi_h u\|_{L^2(L^2)}\,\|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)}\,. \end{aligned}$$

Finally substituting the estimates for S_1, S_2, S_3 and S_4 in (3.41) we find that

$$\|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)} \le Ch\left(\|u - \Pi_h u\|_{L^2(L^2)} + \|\partial_t(u - \Pi_h u)\|_{L^2(L^2)}\right). \tag{3.44}$$

To estimate the third term of (3.36), we follow similar arguments as used in the proof of Theorem 2.7.7 of Chapter 2. We apply coercivity condition (3.8) for $\Pi_h u - u_h \in U_{h,ad} \subset \mathcal{U}$, utilize the discrete variational inequality and projection property of Π_h alongwith result (3.32), estimates of Lemmas 3.3.1 and 3.3.4 and above relation (3.44) to obtain

$$\|y_h(\Pi_h u) - y_h\|_{L^2(L^2)} \leq Ch\left(\|u - \Pi_h u\|_{L^2(L^2)} + \|\partial_t(u - \Pi_h u)\|_{L^2(L^2)}\right). \tag{3.45}$$

The proof follows by inserting (3.44) and (3.45) in (3.36) and using the estimates of Lemma 3.3.4 and approximation properties of Π_h. The h^2 convergence of state error implies the h^2 convergence of costate error. □

In mesh dependent norm:

In order to obtain the estimates for $\|y - y_h\|_{L^2(V(h))}$ and $\|p - p_h\|_{L^2(V(h))}$, we apply triangle inequality, estimates of Lemma 3.3.1 and Lemma 3.3.3 to obtain

$$\|y - y_h\|_{L^2(V(h))} \leq \|y - y_h(u)\|_{L^2(V(h))} + \|u - u_h\|_{L^2(L^2)} \tag{3.46}$$

$$\|p - p_h\|_{L^2(V(h))} \leq \|p - p_h(y)\|_{L^2(V(h))} + \|y - y_h\|_{L^2(L^2)}. \tag{3.47}$$

Theorem 3.3.12. *Let u be a local optimal control of problem (3.4) with the associated state y and costate p, respectively, and let u_h, y_h and p_h be the solution of the semi discrete optimal control problem (3.9)-(3.10), then the following discretization error estimates in energy norm are satisfied*

$$\|y - y_h\|_{L^2(V(h))} + \|p - p_h\|_{L^2(V(h))} \leq Ch.$$

The proof follows by inserting the estimates of $\|u - u_h\|_{L^2(L^2)}$ and $\|y - y_h\|_{L^2(L^2)}$ in (3.46) and (3.47), respectively.

3.4 Error estimates for the fully-discrete scheme

In this Section, we will analyze the error estimates of the fully discrete DFV approximation. The most of the tools here are carried out from semi-discrete analysis.

For this case, we start by assuming $y_h^m(\tilde{u})$ and $p_h^{m-1}(\tilde{y})$ to be the solutions of the following equations:

$$
\begin{aligned}
(\partial_t y_h^m(\tilde{u}), \gamma v_h) + A_h(y_h^m(\tilde{u}), v_h) + (\varphi(y_h^m(\tilde{u})), \gamma v_h) &= (\mathcal{B}\tilde{u}^m + f^m, \gamma v_h), \\
\forall v_h \in V_h;\ m = 1, ..., M; \quad y_h^0(\tilde{u})(x) &= y_{0,h},\ x \in \Omega, \quad (3.48) \\
-(\partial_t p_h^m(\tilde{y}), \gamma q_h) + A_h(p_h^{m-1}(\tilde{y}), q_h) + (\varphi'(y^m) p_h^{m-1}(\tilde{y}), \gamma q_h) &= (\tilde{y}^m - y_d^m, \gamma q_h), \\
\forall q_h \in V_h;\ m = M, ..., 1; \quad p_h^M(\tilde{y})(x) &= 0, \quad x \in \Omega. \quad (3.49)
\end{aligned}
$$

We define the following discrete time-dependent norms to be used for our further analysis

$$
\|v\|_{L^2(L^2)} := \left(\sum_{m=1}^{M} k_m \|v^m\|_{0,\Omega}^2 \right)^{1/2}, \quad \|v\|_{L^\infty(L^2)} := \max_{1 \le m \le M} \|v^m\|_{0,\Omega}.
$$

Similarly we denote time and mesh dependent norms as

$$
\|v\|_{L^2(V(h))} := \left(\sum_{m=1}^{M} k_m \|v^m\|_h^2 \right)^{1/2}, \quad \|v\|_{L^\infty(V(h))} := \max_{1 \le m \le M} \|v^m\|_h.
$$

Let $y_h = (y_h^0, y_h^1, .., y_h^M), p_h = (p_h^0, p_h^1, .., p_h^M)$ and $u_h = (u_h^0, u_h^1, .., u_h^M)$. For clarity we note that $y_h^m = y_h^m(u_h)$, $p_h^m = p_h^m(y_h)$ and $p_h^m(u) = p_h^m(y_h(u))$. Then we establish the following intermediate lemma for $\tilde{u} = u$ and $\tilde{y} = y$.

Lemma 3.4.1. *Let $y_h^n(u)$ and $p_h^n(y)$ be the solutions of auxiliary equations (3.48) and (3.49), respectively. Then for sufficiently small k, there exists a positive constant C independent of h and k such that*

$$
\|y_h^n(u) - y_h^n\|_h \le C \|u - u_h\|_{L^2(L^2)}, \quad \|p_h^n(y) - p_h^n\|_h \le C \|y - y_h\|_{L^2(L^2)}.
$$

Proof. Subtracting equations (3.12) from (3.48), denoting $y_h^m(u) - y_h^m = \eta^m$ and choosing $v_h = \partial_t \eta^m$ in the discrete equations, we can obtain the following equation:

$$
(\partial_t \eta^m, \gamma \partial_t \eta^m) + A_h(\eta^m, \partial_t \eta^m) = (\mathcal{B}(u^m - u_h^m), \gamma \partial_t \eta^m) + (\varphi(y_h^m) - \varphi(y_h^m(u)), \gamma \partial_t \eta^m).
$$

Utilizing the definitions of the norm $\|\cdot\|_0$ and $\epsilon_a(\cdot, \cdot)$ in the above equation, we can express it as

$$\|\partial_t \eta^m\|_0^2 + a_h(\eta^m, \partial_t \eta^m) =(\mathcal{B}(u^m - u_h^m), \gamma \partial_t \eta^m) + \epsilon_a(\eta^m, \partial_t \eta^m) \\ + (\varphi(y_h^m) - \varphi(y_h^m(u)), \gamma \partial_t \eta^m). \tag{3.50}$$

The estimate (2.19) and inverse inequality together with Young's inequality implies

$$\epsilon_a(\eta^m, \partial_t \eta^m) \leq C(\epsilon) \|\eta^m\|_h^2 + \epsilon \|\partial_t \eta^m\|_{0,\Omega}^2. \tag{3.51}$$

From the Cauchy-Schwarz inequality and relation (2.10), we can obtain

$$(\mathcal{B}(u^m - u_h^m), \gamma \partial_t \eta^m) \leq C(\epsilon) \|u^m - u_h^m\|_{0,\Omega}^2 + \epsilon \|\partial_t \eta^m\|_{0,\Omega}^2. \tag{3.52}$$

Applying the Lipschitz continuity of $\varphi(\cdot)$ together with (2.17) we find that

$$(\phi(y_h^m) - \phi(y_h^m(u)), \gamma \partial_t \eta^m) \leq C(\epsilon) \|\eta^m\|_h^2 + \epsilon \|\partial_t \eta^m\|_{0,\Omega}^2. \tag{3.53}$$

A simple manipulation shows that

$$a_h(\eta^m, \partial_t \eta^m) \geq \frac{1}{2k_i}(a_h(\eta^m, \eta^m) - a_h(\eta^{m-1}, \eta^{m-1})). \tag{3.54}$$

Inserting the bounds (3.51), (3.52) and (3.53)with appropriate value of ϵ in relation (3.50), we get

$$a_h(\eta^m, \eta^m) - a_h(\eta^{m-1}, \eta^{m-1}) \leq C\left(2k_m \|\eta^m\|_h^2 + 2k_m \|u^m - u_h^m\|_{0,\Omega}^2\right),$$

where we have used the equivalence of $\|\cdot\|_0$ and $\|\cdot\|_{0,\Omega}$ and the inequality (3.54). Summing m from 1 to n, using coercivity of $a_h(\cdot,\cdot)$ and noting $\eta^0 = 0$, we find that

$$\|\eta^n\|_h^2 \leq C\{\sum_{m=1}^{n} k_m \|\eta^m\|_h^2 + \sum_{m=1}^{n} k_m \|u^m - u_h^m\|_{0,\Omega}^2\},$$

and an appeal to the discrete Gronwall's inequality implies that

$$\|y_h^n(u) - y_h^n\|_h \leq C \|u - u_h\|_{L^2(L^2)}.$$

For estimating $\|p_h^n(y) - p_h^n\|_h$, we proceed similarly as we have estimated $\|y_h^n(u) - y_h^n\|_h$.

On subtracting (3.13) from (3.49), writing $p_h^m(y) - p_h^m = \mu^m$ and choosing $q_h = \partial_t \mu^m$, we infer that for $m = 1, 2, ..., M$

$$\begin{aligned}\|\partial_t \mu^m\|_0^2 - a_h(\mu^{m-1}, \partial_t \mu^m) =& (y_h^m - y^m, \gamma \partial_t \mu^m) - \epsilon_a(\mu^{m-1}, \partial_t \mu^m) \\ &+ (\varphi'(y^m) p_h^{m-1}(y) - \varphi'(y_h^m) p_h^{m-1}, \gamma \partial_t \mu^m).\end{aligned} \tag{3.55}$$

We note that the following relation holds

$$-a_h(\mu^{m-1}, \partial_t \mu^m) \geq \frac{1}{2k_i}(a_h(\mu^{m-1}, \mu^{m-1}) - a_h(\mu^m, \mu^m)). \tag{3.56}$$

It follows from the assumption $\varphi'(\cdot) \geq 0$ and Lipschitz continuity that

$$(\varphi'(y^m) p_h^{m-1}(y) - \varphi'(y_h^m) p_h^{m-1}, \gamma \partial_t \mu^m) \leq C(\epsilon) \left\|\mu^{m-1}\right\|_h^2 + \epsilon \|\partial_t \mu^m\|_{0,\Omega}^2. \tag{3.57}$$

Also, using the arguments used in derivation of inequalities (3.51) and (3.53), we have the following bounds

$$\epsilon_a(\mu^{m-1}, \partial_t \mu^m) \leq C(\epsilon) \left\|\mu^{m-1}\right\|_h^2 + \epsilon \|\partial_t \mu^m\|_{0,\Omega}^2, \tag{3.58}$$

$$(y_h^m - y^m, \gamma(\partial_t \mu^m)) \leq C(\epsilon) \|y^m - y_h^m\|_{0,\Omega}^2 + \epsilon \|\partial_t \mu^m\|_{0,\Omega}^2. \tag{3.59}$$

On plugging the estimates (3.56), (3.57), (3.58), (3.59) in relation (3.55) for appropriate value of ϵ gives

$$a_h(\mu^{m-1}, \mu^{m-1}) - a_h(\mu^m, \mu^m) \leq C\left(2k_m \left\|\mu^{m-1}\right\|_h^2 + 2k_m \|y^m - y_h^m\|_{0,\Omega}^2\right).$$

Summing m from $n+1$ to M, using coercivity of $a_h(\cdot, \cdot)$ and noticing $\mu^M = 0$ in the above relation, we obtain

$$\|\mu^n\|_h^2 \leq C\{\sum_{m=n+1}^{M} k_m \|\mu^m\|_h^2 + \sum_{m=n+1}^{n} k_m \|y^m - y_h^m\|^2\}$$

The proof follows from discrete Gronwall's inequality for sufficiently small k_m. □

With the help of elliptic projection R_h defined in (3.26) and choosing $y_h^0 = R_h y_0(x)$ for $x \in \Omega$, for a given $\tilde{u}$, the following estimates can also be derived for the fully-discrete case by appealing to the arguments used for the standard DFV analysis for parabolic problems. Therefore, we refrain ourself for providing this proof and we refer to [9].

Lemma 3.4.2. *For any $\tilde{u} \in L^2(L^2)$, $\tilde{y} = y(\tilde{u}) \in L^2(H^2)$, there exists a positive constant C independent of h and k such that*

$$\|y(\tilde{u}) - y_h(\tilde{u})\|_{L^2(V(h))} + \|p(\tilde{y}) - p_h(\tilde{y})\|_{L^2(V(h))} \leq C(h + k).$$

Lemma 3.4.3. *For any $\tilde{u} \in L^2(L^2)$ and $\tilde{y} = y(\tilde{u}) \in L^2(H^3)$, there exists a positive constant C independent of h and k such that*

$$\|y(\tilde{u}) - y_h(\tilde{u})\|_{L^2(L^2)} + \|p(\tilde{y}) - p_h(\tilde{y})\|_{L^2(L^2)} + \|p(\tilde{u}) - p_h(\tilde{u})\|_{L^2(L^2)} \leq C(h^2 + k),$$

and in particular, for $\tilde{u} = u_h$, we have

$$\|p(u_h) - p_h(u_h)\|_{L^2(L^2)} \leq C(h^2 + k). \tag{3.60}$$

3.4.1 Error estimates for control

With variational discretization:

Now, we prove the following result.

Theorem 3.4.4. *Let u be a fixed local optimal control of problem (3.4) and u_h^m be the solution of the fully-discrete optimal control problem (3.12)-(3.14) at $t = t_m$ with variational discretization approach, then the following error estimate holds.*

$$\|u - u_h\|_{L^2(L^2)} \leq C(h^2 + k).$$

Proof. For each time interval I_m the continuous variational inequality will be of the form

$$(\lambda u^m + \mathcal{B}^* p^{m-1}, \tilde{u} - u^m) \geq 0, \quad \forall \tilde{u} \in U_{ad} \tag{3.61}$$

and the discrete variational inequality is

$$(\lambda u_h^m + \mathcal{B}^* p_h^{m-1}, \tilde{u}_h - u_h^m) \geq 0, \forall \tilde{u}_h \in U_{ad}. \tag{3.62}$$

Choosing $\tilde{u} = u_h^m$ in (3.61) and $\tilde{u}_h = u^m$ in (3.62), we have

$$(\lambda u^m + \mathcal{B}^* p^{m-1}, u_h^m - u^m) \geq 0 \leq -(\lambda u_h^m + \mathcal{B}^* p_h^{m-1}, u^m - u_h^m).$$

The coercivity condition (3.8) for $u - u_h \in U_{ad} \subset L^2(L^2)$ yields the result

$$C \|u - u_h\|^2_{L^2(L^2)} \leq \|p(u_h) - p_h\|_{L^2(L^2)} \|u - u_h\|_{L^2(L^2)},$$

and the proof follows by applying the estimate (3.60) in the above relation. □

With piecewise constant discretization:

Theorem 3.4.5. *Let u be a fixed local optimal control of problem (3.4) and u_h^m be the solution of the fully-discrete optimal control problem (3.12)-(3.13) at $t = t_m$ with piecewise constant discretization of controls, then we have the following discretization error estimate*

$$\|u - u_h\|_{L^2(L^2)} \leq C(h + k)$$

is fulfilled.

Proof. For this case we use the second order sufficient condition (3.8) and the relations

$$(\lambda u_h^m + \mathcal{B}^* p_h^{m-1}, \Pi_0 u^m - u_h^m) \geq 0 \geq (\lambda u^m + \mathcal{B}^* p^{m-1}, u^m - u_h^m).$$

to obtain the following estimates

$$\begin{aligned} C \|u - u_h\|^2_{L^2(L^2)} \leq & \sum_{m=1}^{M} k_m (\mathcal{B}^* p_h^{m-1} - \mathcal{B}^* p^{m-1}(u_h), u^m - u_h^m) \\ & + \sum_{m=1}^{M} k_m (\lambda u_h^m + \mathcal{B}^* p_h^{m-1}, \Pi_0 u^m - u^m). \end{aligned} \tag{3.63}$$

The terms in the above relation is bounded by using (3.60), continuity of operator $\mathcal{B}$, projection property of Π_0 and uniform boundedness of p_h. The proof is completed by inserting the bounds of (3.63). □

With piecewise linear discretization:

The following result for control error follows from the same idea used in the proof of Theorem 3.3.9. Therefore, we present here outline of the proof of the following Theorem.

Theorem 3.4.6. *Let u be a fixed local optimal control of problem (3.4) and u_h^m be the solution of the fully-discrete optimal control problem (3.12)-(3.13) at $t = t_m$ with piecewise linear discretization of controls then the following estimate holds true*

$$\|u - u_h\|_{L^2(L^2)} \leq C(h^{3/2} + k).$$

Proof. First by testing the continuous and discrete variational inequalities on each subinterval I_m with $u_h^m \in U_{h,ad} \subset U_{ad}$ and $\tilde{u}_h^m \in U_{h,ad}$, we find that

$$(\lambda u_h^m + \mathcal{B}^* p_h^m, \tilde{u}_h^m - u_h^m) \geq 0 \geq (\lambda u^m + \mathcal{B}^* p^m, u^m - u_h^m).$$

Using the condition (3.8) for $u - u_h \in U_{h,ad}$ and the above relation, we can obtain

$$\begin{aligned} C\|u - u_h\|_{L^2(L^2)}^2 \leq \sum_{m=1}^{M} k_m [&\lambda(u^m - u_h^m, u^m - \tilde{u}_h^m) + (p^{m-1} - p_h^{m-1}, \mathcal{B}(u^m - \tilde{u}_h^m)) \\ &+ (p_h^{m-1} - p^{m-1}(u_h), \mathcal{B}(u^m - u_h^m)) + (\lambda u^m + \mathcal{B}^* p^{m-1}, \tilde{u}_h^m - u^m)], \end{aligned}$$

which on applying Cauchy-Schwarz inequality and estimates of Lemma 3.3.8 and (3.60) implies that

$$\begin{aligned} C\|u - u_h\|_{L^2(L^2)}^2 \leq &\Big(\|u - u_h\|_{L^2(L^2)} + \|p - p_h\|_{L^2(L^2)}\Big)\|u - \tilde{u}_h\|_{L^2(L^2)} \\ &+ C(h^2 + k)\|u - u_h\|_{L^2(L^2)} + \frac{C}{\lambda} h^3 \|\nabla p\|_{L^2(L^\infty)}^2. \end{aligned} \tag{3.64}$$

Now, we can write $\|u - \tilde{u}_h\|_{L^2(L^2)} = \sum_{m=1}^{M} k_m \|u^m - \tilde{u}_h^m\|_{0,\Omega}$. Similar to the semidiscrete case, for each $m = 1, 2, ..M$, we can obtain the estimate $\|u^m - \tilde{u}_h^m\|_{0,\Omega}^2 \leq \frac{C}{\lambda^2} h^4 \|\nabla^2 p^m\|_{0,\Omega}^2 + \frac{C}{\lambda^2} h^3 \|\nabla p^m\|_{L^\infty(\Omega)}^2$ by using the definition (3.31) and disjoint sets $\mathcal{T}_{h,m}^1$, $\mathcal{T}_{h,m}^2$ and $\mathcal{T}_{h,m}^3$ together with assumption (3.3.7) and property (3.7). The required result follows by inserting the estimates of the terms of (3.64), applying Young's inequality and following similar arguments as in the semi-discrete case. □

3.4.2 Error estimates for state and costate

The estimate for state and costate error for fully-discrete case can be obtained by following the similar arguments used in the semi-discrete case. Therefore, we state the following Theorem which gives the estimate for the errors $\|y - y_h\|_{L^2(L^2)}$ and $\|p - p_h\|_{L^2(L^2)}$ and here we skip the proof.

Theorem 3.4.7. *Let u be an optimal control of problem (3.4) with the associated state y and costate p, respectively, and let u_h^m, y_h^m and p_h^m be the solution of the fully-discrete optimal control problem (3.12)-(3.13) at $t = t_m$, then the following discretization error estimates are satisfied*

$$\|y - y_h\|_{L^2(L^2)} + \|p - p_h\|_{L^2(L^2)} \leq C(h^2 + k).$$

In mesh-dependent norm:

The following Theorem states the error estimates for state and costate error in discrete norm.

Theorem 3.4.8. *Let u be a local optimal control of problem (3.4) with the associated state y and costate p, respectively, and let u_h^m, y_h^m and p_h^m be the solution of the fully-discrete optimal control problem (3.12)-(3.13) at $t = t_m$, then the following discretization error estimates in energy norm are satisfied*

$$\|y - y_h\|_{L^2(V(h))} + \|p - p_h\|_{L^2(V(h))} \leq C(h + k).$$

Proof. On applying triangle inequality, results of Lemma 3.4.1 and Lemma 3.4.2 we find that

$$\|y - y_h\|_{L^2(V(h))} \leq \|y - y_h(u)\|_{L^2(V(h))} + \|u - u_h\|_{L^2(L^2)} \tag{3.65}$$

$$\|p - p_h\|_{L^2(V(h))} \leq \|p - p_h(y)\|_{L^2(V(h))} + \|y - y_h\|_{L^2(L^2)} . \tag{3.66}$$

The required convergence results can be readily obtained by putting the estimates of $\|u - u_h\|_{L^2(L^2)}$ and $\|y - y_h\|_{L^2(L^2)}$ in (3.65) and (3.66), respectively. □

3.5 Numerical Experiments

In this Section, we present our numerical results to validate the theoretical error estimates derived for control, state and costate variables and to illustrate the performance of the proposed method.

Implementation aspects

For computational aspects we have used the idea of interpolated coefficients to approximate the nonlinear term. This idea also used to solve nonlinear heat equations, see (chen89). For the time discretization, we have employed a first order backward Euler formula with a fixed time step. The resulting non-linear system of equations is solved by applying Newton method. Then on utilizing the idea of interpolated coefficients, the discrete state equation (3.12) for $m = 1, ..., M$ can be reformulated as

$$(\partial_t y_h^m, \gamma v_h) + A_h(y_h^m, v_h) + (I_h \varphi(y_h^m), \gamma v_h) = (\mathcal{B} u_h^m + f^m, \gamma v_h). \tag{3.67}$$

Applying backward Euler scheme in (3.67) leads to a nonlinear system of equations

$$\mathbb{M}\frac{Y^m - Y^{m-1}}{k} + \mathbb{A}Y^m + \mathbb{M}\varphi(Y^m) = \mathbb{G}U^m + \mathbb{F}^m. \tag{3.68}$$

where the matrix blocks $\mathbb{A}$, $\mathbb{M}$, $\mathbb{G}$ and vector $\mathbb{F}$ is same as defined in Section 2.8 of Chapter 2. A direct computation shows that Jacobian matrix is $\mathbb{J} = (\mathbb{M} + k\mathbb{A}) + k\mathbb{M}_{\varphi'}$ where $\mathbb{M}_{\varphi'}$ is the mass matrix defined as $\mathbb{M}_{\varphi'} = \int\limits_{\Omega} \frac{\partial \varphi^m}{\partial y^m} \Phi_i \Phi_j^* = \{m_{ij}\varphi'(Y_j^m)\}_{N\times N}$. Here also, we observe that Jacobian matrix can be obtained by simply multiplying m_{ij} by $\varphi'(Y_j^m)$ which is an advantage of using interpolated coefficient method.

In order to verify the theoretical convergence results, we will consider the optimal control problem (3.1)-(3.2) with domain $\Omega = (0, 1)^2$ and final time $T = 1$ in the numerical examples.

Example 3.5.1. The data are as follows:

$$u_a = 0,\ u_b = 1, \quad \lambda = 0.5, \quad \varphi(y) = y^3, \quad \mathcal{A} = \begin{pmatrix} 1+x_1^2 & 0 \\ 0 & 1+x_2^2 \end{pmatrix}$$

$$y(t,x) = e^t x_1 x_2 (x_1 - 1)(x_2 - 1),\ p(t,x) = (e^t - e) x_1 x_2 (x_1 - 1)(x_2 - 1),$$

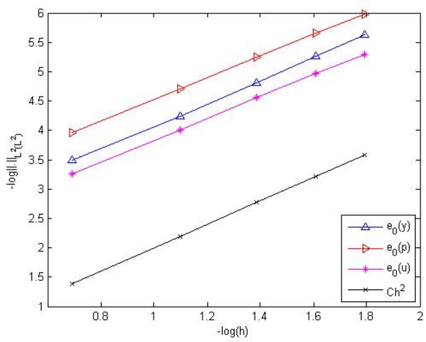
$$u(t,x) = max(0, min(1, -\frac{1}{\lambda} p(t,x)).$$

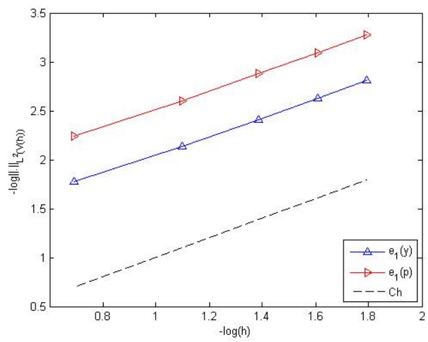
The source term f and the desired state y_d is of the form

$$f(t,x) = \partial_t y(t,x) - \nabla \cdot \mathcal{A} \nabla y(t,x) + \varphi(y(t,x)) - u(t,x),$$

$$y_d(t,x) = y(t,x) + \partial_t p(t,x) + \nabla \cdot \mathcal{A} \nabla p(t,x) - \varphi'(y(t,x)) p(t,x).$$

We have used the algorithm 1 defined in Chapter 2 to solve optimal control problem with the discretization described in Section 3.2. We will use the following notations





(a) Convergence of state, costate and control error in discrete $L^2(L^2)$-norm.

(b) Convergence of state and costate error in discrete $L^2(V(h))$-norm .

Figure 3.1: The convergence rates of the DFV approximations of the state, adjoint state and control variables with variational discretization approach under the refinement of the spatial triangulation for time step size $k = 0.01$.

to measure the errors in discrete $L^2(L^2)$-norm for optimal state, costate and control variables and corresponding observed rates:

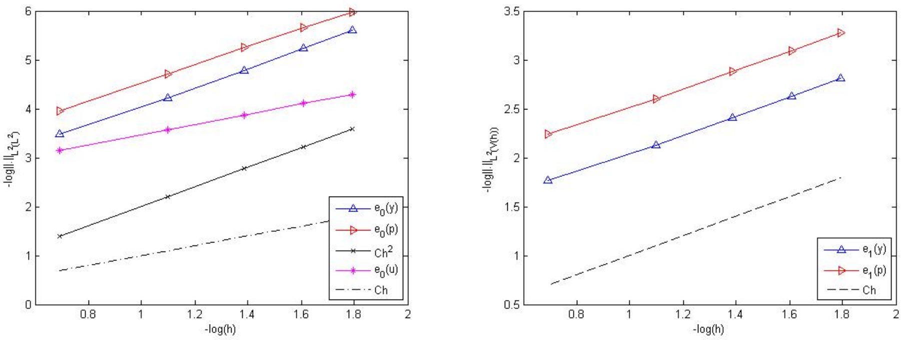
$$\left.\begin{aligned} e_0(y) &:= \|y - y_h\|_{L^2(L^2)}\,, \quad & r_0(y) &:= \frac{\log(e_0(y)/\hat{e}_0(y))}{\log(h/\hat{h})}, \\ e_0(p) &:= \|p - p_h\|_{L^2(L^2)}\,, \quad & r_0(p) &:= \frac{\log(e_0(p)/\hat{e}_0(p))}{\log(h/\hat{h})}, \\ e_0(u) &:= \|u - u_h\|_{L^2(L^2)}\,, \quad & r_0(u) &:= \frac{\log(e_0(u)/\hat{e}_0(u))}{\log(h/\hat{h})}. \end{aligned}\right\} \tag{3.69}$$

Similarly, we denote the state and costate errors in discrete $L^2(V(h))$-norm and corresponding observed rates by

$$\left.\begin{aligned} e_1(y) &:= \|y - y_h\|_{L^2(V(h))}\,, \qquad & r_1(y) &:= \frac{\log(e_1(y)/\hat{e}_1(y))}{\log(h/\hat{h})}, \\ e_1(p) &:= \|p - p_h\|_{L^2(V(h))}\,, \qquad & r_1(p) &:= \frac{\log(e_1(p)/\hat{e}_1(p))}{\log(h/\hat{h})}. \end{aligned}\right\} \tag{3.70}$$

Here, e and $\hat{e}$ represent computed errors on two consecutive meshes of length h and $\hat{h}$, respectively.

The penalty parameters are set as $\alpha = 10$, $\beta = 1$ and $\theta = -1$.

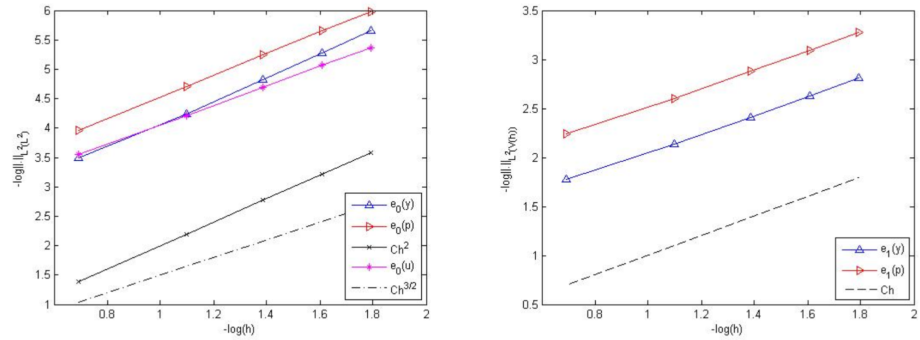


(a) Convergence of state, costate and control error in discrete $L^2(L^2)$-norm.

(b) Convergence of state and costate error in discrete $L^2(V(h))$-norm .

Figure 3.2: The convergence rates of the DFV approximations of the state, costate and control variables using piecewise constant discretization of control under the refinement of the spatial triangulation for time step size $k = 0.01$.

For a fixed time step $k = 0.01$, the convergence behaviour of state, costate and control errors are depicted in Figure 3.1 for variational discretization approach, in Figure



(a) Convergence of state, costate and control error in discrete $L^2(L^2)$-norm.

(b) Convergence of state and costate error in discrete $L^2(V(h))$-norm .

Figure 3.3: The convergence rates of the DFV approximations of the state, adjoint state and control variables with piecewise linear discretization of control under the refinement of the spatial triangulation for time step size $k = 0.01$.

3.2 for piecewise constant control discretization approach and in Figure 3.3 for piecewise linear discretization approach. The computed approximation errors and respective convergence rates for state, costate and control variables in suitable discrete norms for three different control discretization approaches have been reported in Table 3.1. The computed convergence rates are in agreement with the theoretical results.

Now, we illustrate the performance of the proposed method by considering the following concretion of the model problem (3.1)-(3.2) in which exact analytical solutions are not known.

Example 3.5.2. The problem represents nonstationary heating of a body $\Omega = (0,1)^2$ with intial temperature zero. The given data are:

$$u_a = -0.25,\, u_b = 0.25, \quad \lambda = 1, \quad \varphi(y) = y^3, \quad \mathcal{A} = I_d,$$

where I_d is the 2×2 identity matrix. The source term f and the desired temperature y_d

h	$e_0(y)$	$r_0(y)$	$e_1(y)$	$r_1(y)$	$e_0(p)$	$r_0(p)$	$e_1(p)$	$r_1(p)$	$e_0(u)$	$r_0(u)$
Variational discretization approach										
0.5000	0.0305	-	0.1697	-	0.0190	-	0.1058	-	0.0381	-
0.3333	0.0143	1.8633	0.1184	0.8883	0.0090	1.8368	0.0739	0.8843	0.0181	1.8368
0.2500	0.0081	1.9739	0.0899	0.9572	0.0052	1.8894	0.0562	0.9519	0.0105	1.8889
0.2000	0.0052	2.0076	0.0721	0.9842	0.0034	1.8456	0.0452	0.9781	0.0069	1.8460
0.1666	0.0035	2.0233	0.0601	0.9967	0.0025	1.7648	0.0377	0.9896	0.0050	1.7648
Piecewise constant control										
0.5000	0.0308	-	0.1701	-	0.0190	-	0.1058	-	0.0427	-
0.3333	0.0146	1.8447	0.1186	0.8891	0.0090	1.8381	0.0739	0.8843	0.0280	1.0377
0.2500	0.0083	1.9594	0.0900	0.9586	0.0052	1.8901	0.0562	0.9519	0.0207	1.0514
0.2000	0.0053	1.9945	0.0722	0.9857	0.0034	1.8465	0.0452	0.9780	0.0164	1.0415
0.1666	0.0036	2.0101	0.0602	0.9980	0.0025	1.7656	0.0377	0.9896	0.0136	1.0301
Piecewise linear control										
0.5000	0.0305	-	0.1697	-	0.0190	-	0.1058	-	0.0287	-
0.3333	0.0143	1.8662	0.1183	0.8885	0.0090	1.8366	0.0739	0.8843	0.0149	1.6121
0.2500	0.0080	2.0041	0.0898	0.9587	0.0052	1.8868	0.0562	0.9518	0.0091	1.6901
0.2000	0.0051	2.0254	0.0721	0.9841	0.0034	1.8442	0.0452	0.9780	0.0062	1.6945
0.1666	0.0035	2.0683	0.0601	0.9971	0.0025	1.7615	0.0377	0.9895	0.0046	1.6311

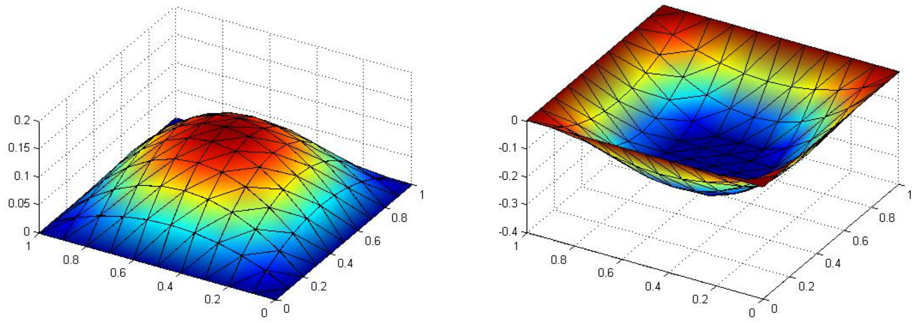
Table 3.1: Numerical results for state, costate and control errors with $k = 0.01$ on a sequence of uniformly refined partition of $\Omega = (0, 1)^2$.

are given by

$$\begin{aligned} f =& 5(x_1^2 - x_1)(x_2^2 - x_2) - 2(x_1^2 - x_1 + x_2^2 - x_2)t + 125(x_1^2 - x_1)^3(x_2^2 - x_2)^3 t^3 \\ & - max(-0.25, min(0.25, -5(x_1^2 - x_1)(x_2^2 - x_2)\sin(\pi t))) \quad \text{and} \\ y_d =& 5(x_1^2 - x_1)(x_2^2 - x_2)t + 5\pi(x_1^2 - x_1)(x_2^2 - x_2)\sin(\pi t) + 2(x_1^2 - x_1 + x_2^2 - x_2) \\ & \sin(\pi t) - 375(x_1^2 - x_1)^3(x_2^2 - x_2)^3 t^2 \sin(\pi t), \text{ respectively.} \end{aligned}$$

The computed optimal control u_h acting as a heat source on the body Ω and the temperature y_h at time $t = 0.5$ when $h = 10^{-1}$ and $k = 0.01$ are shown in Figure 3.4.

Moreover, the effect of reducing control cost on the the minimum values of objective functional is listed in Table 3.2. We close this Section by making the following remark.



(a) The numerical solution y_h.

(b) The numerical solution u_h.

Figure 3.4: The computed optimal control u_h and associated optimal state y_h at t=0.5.

λ	0.1	0.01	0.001	0.0001	0.00001
$J(y_h, u_h)$	3.1276736	3.1259664	3.1254841	3.1253762	3.1253623

Table 3.2: The values of objective functional for different regularization parameter for the DFV approximations of the semilinear parabolic optimal control problem.

Remark 3.5.3. *For our numerical experiments, we have considered $\theta = -1$ (SIPG). However, we have observed similar rate of convergence for the other two cases $\theta = 1$ (NIPG) and $\theta = 0$ (IIPG).*

CHAPTER 4

Semilinear hyperbolic optimal control problems

In this Chapter, we extend the analysis of DFV approximations of semilinear parabolic optimal control problems presented in Chapter 3 to the optimal control problems governed by a class of semilinear hyperbolic partial differential equations with control constraints. The spatial discretization of the state and costate variables follows DFV schemes with piecewise linear elements, whereas three different strategies are used for the control approximation: variational discretization, piecewise constant and piecewise linear discretization. Here also, we have employed *optimize then discretize* approach to approximate the control problem. *A priori* error estimates for control, state and costate variables are derived in suitable natural norms. Numerical experiments are presented to illustrate the performance of the proposed scheme and to confirm the predicted accuracy of the theoretical convergence rates.

4.1 Introduction

The hyperbolic optimal control problems arise in medical applications, acoustic problems as noise suppression and for optimal control in linear elasticity (cf.[6, 24, 66]). We consider here the following distributed optimal control problem governed by a semilinear wave equation with control u and state y.

$$\min_{u\in U_{ad}} J(y,u) := \frac{1}{2}\int_0^T \left(\|y(t,x) - y_d(t,x)\|^2_{0,\Omega} + \lambda \|u(t,x)\|^2_{0,\Omega} \right) \mathrm{dt}, \tag{4.1}$$

subject to

$$\left.\begin{aligned} \partial_{tt}y - \nabla\cdot\mathcal{A}\nabla y + \varphi(y) = \mathcal{B}u + f, \ \text{ in } (0,T)\times\Omega, \\ y(t,x) = 0, \ \text{ on } (0,T)\times\partial\Omega, \\ y(0,x) = g(x), \ \partial_t y(0,x) = w(x), \quad x\in\Omega. \end{aligned}\right\} \tag{4.2}$$

Here, the set of admissible controls U_{ad} is same as defined in (3.3) for parabolic case. The proposed model problem describes the optimal vibrations in a bridge with Ω as the domain of the bridge. The overall idea is to identify an additional force u acting in vertical direction giving rise to a transversal displacement y which best approximates the desired evolution y_d of transversal vibrations. We impose the similar assumptions made in Chapter 3 on the given data and nonlinear term for our analysis.

With the introduction of control-to-state mapping $\mathcal{S}$ with $\mathcal{S}(u) = y$, the problem (4.1)-(4.2) reduces to:

$$\min_{u \in U_{ad}} j(u) := \min_{u \in U_{ad}} J(\mathcal{S}(u), u). \tag{4.3}$$

Under some extra assumptions, the problem (4.3) exhibits at least one optimal control with associated state $y = G(u)$ (for details, see [54, 67]). Due to nonlinearity of control-to-state operator the reduced objective functional need not be convex and hence the solutions may not be unique. Therefore, we will use the notion of local solution in the sense of $L^2(L^2)$.

A local solution u of (4.3) in the sense of Definition 3.1.1 satisfies the standard first order necessary optimality condition which can be formulated with the help of the following variational inequality:

$$j'(u)(\tilde{u} - u) \geq 0, \quad \forall \tilde{u} \in U_{ad},$$

and can be further rewritten in the form

$$(\lambda u + \mathcal{B}^* p, \tilde{u} - u)_{L^2(L^2)} \geq 0, \quad \forall \tilde{u} \in U_{ad}. \tag{4.4}$$

Above $(\cdot,\cdot)_{L^2(L^2)} := \int_0^T (\cdot,\cdot) d\tau$ stands for the scalar product in $L^2(L^2)$ with the associated norm $\|\cdot\|_{L^2(L^2)}$ as defined in the previous Chapter. We will use these notations frequently throughout this paper. In (4.4), the function p is the costate associated to local control u and solves the costate equation given by

$$\left.\begin{aligned} \partial_{tt} p - \nabla \cdot \mathcal{A} \nabla p + \varphi'(y) p = y - y_d, \ \text{in } (0,T) \times \Omega, \\ p(t,x) = 0, \ \text{on } (0,T) \times \partial\Omega, \\ p(T,x) = 0, \ \partial_t p(T,x) = 0, \quad x \in \Omega. \end{aligned}\right\}$$

We stress that with some extra regularity on the solution operator $\mathcal{S}$ the local solution $u \in U_{ad}$ also satisfies the following second order sufficient optimality condition. This assumption seems to be legitimate, for details we refer to [67] (see also [19]).

$$\exists\, C > 0 \,:\, j''(u)(\tilde{u}, \tilde{u}) \geq C \, \|\tilde{u}\|^2_{L^2(L^2)}, \quad \forall \tilde{u} \in L^2(L^2). \tag{4.5}$$

The rest part of this Chapter is arranged in the following way. In Section 4.2, we obtain discrete formulation of the semilinear hyperbolic optimal control problem (4.1)-(4.2) by applying DFV scheme with three different control discretization techniques mentioned in previous Chapters. Section 4.3 deals with *a priori* error estimates for different types of control discretization. Further in Section 4.4, we apply an implicit difference scheme to approximate the time derivative and obtain fully-discrete DFV formulation of hyperbolic control problem. Therein, we present the convergence results of control, state and costate errors with fully-discrete scheme. Finally, in Section 4.5, we present numerical experiments to illustrate the theoretical results and performance of the method.

4.2 Discretization

In this Section, we will approximate the continuous optimal system directly by applying the piecewise linear DFV schemes with three different control discretization (variational discretization, piecewise linear and constant discretization) techniques. We first apply DFV methods presented in Section 2.2 of Chapter 2 for spatial discretization of the optimal control problem.

4.2.1 Discontinuous finite volume scheme

On applying DFV scheme to discretize the state and costate equations directly, the semidiscrete formulation of semilinear hyperbolic optimal control problem is given by

: Find $(y_h(t,\cdot), p_h(t,\cdot), u_h(t,\cdot)) \in V_h \times V_h \times U_{h,ad}$ with $0 < t \leq T$ such that

$$(\partial_{tt} y_h, \gamma v_h) + A_h(y_h, v_h) + (\varphi(y_h), \gamma v_h) = (\mathcal{B}u_h + f, \gamma v_h), \quad \forall v_h \in V_h, \tag{4.6}$$
$$y_h(0,x) = g_h(x), \quad \partial_t y_h(0,x) = w_h(x), \quad x \in \Omega,$$
$$(\partial_{tt} p_h, \gamma q_h) + A_h(p_h, q_h) + (\varphi'(y_h) p_h, \gamma q_h) = (y_h - y_d, \gamma q_h), \quad \forall q_h \in V_h, \tag{4.7}$$
$$p_h(T,x) = 0, \quad \partial_t p_h(T,x) = 0, \quad x \in \Omega,$$
$$(\lambda u_h + \mathcal{B}^* p_h, \tilde{u}_h - u_h)_{L^2(L^2)} \geq 0, \quad \forall \tilde{u}_h \in U_{h,ad}, \tag{4.8}$$

where, g_h and w_h are certain approximations of $g(x)$ and $w(x)$ to be defined later and the bilinear form $A_h(\cdot,\cdot) : V_h \times V_h \longrightarrow \mathbb{R}$ is defined in (2.15).

4.3 Error estimates

In this Section, we provide *a priori* error estimates for the optimal control problem, in context of fixed local reference solution of the problem (4.3) which fulfills the first and second order optimality conditions. We will derive the estimates for three different control discretization approaches as mentioned earlier in Section 2.2.2.

For a given arbitrary $\tilde{u} \in L^2(L^2)$ and $\tilde{y} = y(\tilde{u}) \in L^2(H_0^1)$, let $y_h(\tilde{u})$ and $p_h(\tilde{y})$ be the solutions of auxiliary equations

$$(\partial_{tt} y_h(\tilde{u}), \gamma v_h) + A_h(y_h(\tilde{u}), v_h) + (\phi(y_h(\tilde{u})), \gamma v_h) = (B\tilde{u} + f, \gamma v_h), \ \forall v_h \in V_h,$$
$$y_h(\tilde{u})(0,x) = g_h(x), \quad \partial_t y_h(\tilde{u})(0,x) = w_h(x), \quad x \in \Omega, \tag{4.9}$$

and

$$(\partial_{tt} p_h(\tilde{y}), \gamma q_h) + A_h(p_h(\tilde{y}), q_h) + (\phi'(\tilde{y}) p_h(\tilde{y}), \gamma q_h) = (\tilde{y} - y_d, \gamma q_h), \ \forall q_h \in V_h,$$
$$p_h(\tilde{y})(T,x) = 0, \quad \partial_t p_h(\tilde{y})(T,x) = 0, \quad x \in \Omega, \tag{4.10}$$

respectively. To avoid ambiguity, we will be using the following notations: $y_h = y_h(u_h)$, $p_h = p_h(y_h)$ and $p_h(\tilde{u}) = p_h(y_h(\tilde{u}))$. Then we have the following Lemma.

Lemma 4.3.1. *Let $y_h(u)$ and $p_h(y)$ be the solutions of (4.9) and (4.10), respectively. Then there exists a positive constant C independent of h such that the following asser-*

tions hold

$$\|y_h(u) - y_h\|_{L^\infty(V(h))} \le C\,\|u - u_h\|_{L^2(L^2)}\,, \;\; \|p_h(y) - p_h\|_{L^\infty(V(h))} \le C\,\|y - y_h\|_{L^2(L^2)}\,.$$

Proof. On subtracting (4.6) from (4.9), we have the following relation for $v_h \in V_h$

$$\begin{aligned}(\partial_{tt}y_h(u) - \partial_{tt}y_h, \gamma v_h) + A_h(y_h(u) - y_h, v_h) + (\varphi(y_h(u)) - \varphi(y_h), \gamma v_h)& \\ = (\mathcal{B}(u - u_h), \gamma v_h).&\end{aligned}$$

Denoting $y_h(u) - y_h = \vartheta$ and choosing $v_h = \partial_t\vartheta$ in the above equation, we get

$$(\partial_{tt}\vartheta, \gamma\partial_t\vartheta) + A_h(\vartheta, \partial_t\vartheta) + (\varphi(y_h(u)) - \varphi(y_h), \gamma\partial_t\vartheta) = (\mathcal{B}(u - u_h), \gamma\partial_t\vartheta).$$

Using the self-adjoint property of γ and rearranging the terms we can easily obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[(\partial_t\vartheta, \gamma\partial_t\vartheta) + A_h(\vartheta, \vartheta)\right] =&(\mathcal{B}(u - u_h), \gamma\partial_t\vartheta) + \frac{1}{2}[A_h(\partial_t\vartheta, \vartheta) - A_h(\vartheta, \partial_t\vartheta)] \\ &- (\varphi(y_h(u)) - \varphi(y_h), \gamma\partial_t\vartheta).\end{aligned}$$

Integrating above equation from 0 to t and taking into account that $\vartheta(0, x) = 0$ and $\partial_t\vartheta(0, x) = 0$

$$\begin{aligned}|||\partial_t\vartheta|||_0^2 + A_h(\vartheta, \vartheta) =&2\int_0^t(\mathcal{B}(u - u_h), \gamma\partial_t\vartheta)d\tau + \int_0^t[A_h(\partial_t\vartheta, \vartheta) - A_h(\vartheta, \partial_t\vartheta)]d\tau \\ &+ 2\int_0^t(\varphi(y_h) - \varphi(y_h(u)), \gamma\partial_t\vartheta)d\tau.\end{aligned}$$

The equivalence of the norms $|||\cdot|||_0$ and $\|\cdot\|_{0,\Omega}$ and the ellipticity of $A_h(\cdot, \cdot)$ gives

$$\begin{aligned}\|\partial_t\vartheta\|_{0,\Omega}^2 + |||\vartheta|||_h^2 \le&C\int_0^t(\mathcal{B}(u - u_h), \gamma\partial_t\vartheta)d\tau + C\int_0^t[A_h(\partial_t\vartheta, \vartheta) - A_h(\vartheta, \partial_t\vartheta)]d\tau \\ &+ C\int_0^t(\varphi(y_h) - \varphi(y_h(u)), \gamma\partial_t\vartheta)d\tau. \qquad (4.11)\end{aligned}$$

Since $\mathcal{B}$ is a continuous linear operator, the first term of (2.91) can be bounded as

$$\int_0^t (\mathcal{B}(u-u_h),\gamma\partial_t\vartheta)d\tau \le \int_0^t C\,\|u-u_h\|_{0,\Omega}\,\|\partial_t\vartheta\|_{0,\Omega}\,d\tau$$
$$\le C\int_0^t \|u-u_h\|_{0,\Omega}^2\,d\tau + C\int_0^t \|\partial_t\vartheta\|_{0,\Omega}^2\,d\tau. \tag{4.12}$$

Applying the estimate (2.18) and standard inverse estimate we can obtain

$$\int_0^t |A_h(\partial_t\vartheta,\vartheta)-A_h(\vartheta,\partial_t\vartheta)|d\tau \le \int_0^t Ch\,\|\vartheta\|_h\,\|\partial_t\vartheta\|_h\,d\tau \le \int_0^t C\,\|\vartheta\|_h\,\|\partial_t\vartheta\|_{0,\Omega}\,d\tau$$
$$\le C\int_0^t \|\vartheta\|_h^2\,d\tau + C\int_0^t \|\partial_t\vartheta\|_{0,\Omega}^2\,d\tau. \tag{4.13}$$

From the Lipschitz continuity of nonlinear term $\varphi(\cdot)$, the property (2.10) of γ and the inequality (2.17), we can get

$$\int_0^t (\varphi(y_h)-\varphi(y_h(u)),\gamma\partial_t\vartheta)d\tau \le \int_0^t C\,\|\vartheta\|_{0,\Omega}\,\|\gamma\partial_t\vartheta\|_{0,\Omega}\,d\tau \le C\int_0^t \|\vartheta\|_h\,\|\partial_t\vartheta\|_{0,\Omega}\,d\tau$$
$$\le C\int_0^t \|\vartheta\|_h^2\,d\tau + C\int_0^t \|\partial_t\vartheta\|_{0,\Omega}^2\,d\tau. \tag{4.14}$$

Inserting the estimates of (4.12), (4.13) and (4.14) in (4.11) we find that

$$\|\partial_t\vartheta\|_{0,\Omega}^2 + \|\vartheta\|_h^2 \le C\int_0^t \|u-u_h\|_{0,\Omega}^2\,d\tau + C\int_0^t \left(\|\partial_t\vartheta\|_{0,\Omega}^2 + \|\vartheta\|_h^2\right)d\tau. \tag{4.15}$$

The application of Gronwall's Lemma in (4.15) implies

$$\|\partial_t\vartheta\|_{0,\Omega}^2 + \|\vartheta\|_h^2 \le C\int_0^T \|u-u_h\|_{0,\Omega}^2\,d\tau = C\,\|u-u_h\|_{L^2(L^2)}^2$$

which further leads to the first required result

$$\|y_h(u)-y_h\|_{L^\infty(V(h))} \le C\,\|u-u_h\|_{L^2(L^2)}\,.$$

For the second result, we proceed similarly by subtracting (4.7) from (4.10), denoting

$p_h(y) - p_h = \eta$ and choosing $q_h = \partial_t \eta$ to get

$$(\partial_{tt}\eta, \gamma\partial_t\eta) + A_h(\eta, \partial_t\eta) + (\varphi'(y)p_h(y) - \varphi'(y_h)p_h, \gamma\partial_t\eta) = (y - y_h, \gamma\partial_t\eta).$$

Following the similar arguments used previously, we can obtain the relation

$$\begin{aligned}\|\partial_t\eta\|_{0,\Omega}^2 + \|\eta\|_h^2 \leq & C\int_0^T (y - y_h, \partial_t\eta)d\tau + C\int_0^T [A_h(\partial_t\eta, \eta) - A_h(\eta, \partial_t\eta)]d\tau \\ & - C\int_0^t (\varphi'(y)p_h(y) - \varphi'(y_h)p_h, \gamma\partial_t\eta)d\tau.\end{aligned}$$

Using the result (2.18), inverse estimate, boundedness and Lipschitz continuity of φ', we can readily obtain

$$\|\partial_t\eta\|_{0,\Omega}^2 + \|\eta\|_h^2 \leq C\int_0^T \|y - y_h\|_{0,\Omega}^2 \, d\tau + C\int_0^T \left(\|\partial_t\eta\|_{0,\Omega}^2 + \|\eta\|_h^2\right) d\tau,$$

which on application of Gronwall's Lemma yields

$$\|p_h(y) - p_h\|_{L^\infty(V(h))} \leq C \, \|y - y_h\|_{L^2(L^2)} \, .$$

$\square$

For our forthcoming analysis, we would need the following assertion which can be easily proved by using the similar arguments used in the proof of Lemma 2.1 in [48]. Therefore, we provide a sketch of the proof.

Lemma 4.3.2. *There exists a positive constant C independent of h such that the following relation holds:*

$$\|\partial_{tt}(y_h(u) - y_h)\|_{L^\infty(L^2)} \leq C \, \|\partial_t(u - u_h)\|_{L^2(L^2)} \, ,$$

Proof. Differentiating (4.2) with respect to t and multiplying by γv_h, we can obtain the following relation by employing discrete state equation for y_h and $y_h(u)$

$$\begin{aligned}(\partial_{ttt}y_h(u) - \partial_{ttt}y_h, \gamma v_h) + A_h(\partial_t(y_h(u) - y_h), v_h)\\ +(\varphi'(y_h(u))\partial_t y_h(u) - \varphi'(y_h)\partial_t y_h, \gamma v_h) = (\mathcal{B}\partial_t(u-u_h), \gamma v_h).\end{aligned}$$

Denoting $y_h(u) - y_h = \mu$ and choosing $v_h = \partial_{tt}\mu$ in the above equation, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[(\partial_{tt}\mu, \gamma\partial_{tt}\mu) + A_h(\partial_t\mu, \partial_t\mu)\right] =& (\mathcal{B}\partial_t(u-u_h), \partial_{tt}\mu)\\ &+ \frac{1}{2}[A_h(\partial_{tt}\mu, \partial_t\mu) - A_h(\partial_t\mu, \partial_{tt}\mu)]\\ &- (\varphi'(y_h(u))\partial_t y_h(u) - \varphi'(y_h)\partial_t y_h, \gamma\partial_{tt}\mu).\end{aligned}$$

Integrating from 0 to t and using the estimate (2.18) of Lemma 2.2.2 and monotonicity of nonlinear term, we can obtain

$$\|\partial_{tt}\mu\|_{0,\Omega}^2 + \|\partial_t\mu\|_h^2 \leq C\int_0^t \|\partial_t(u-u_h)\|_{0,\Omega}^2\,d\tau + C\int_0^t \left(\|\partial_{tt}\mu\|_{0,\Omega}^2 + \|\partial_t\mu\|_h^2\right)d\tau.$$

Using Gronwall's Lemma we find that

$$\|\partial_{tt}(y_h(u) - y_h)\|_{L^\infty(L^2)} \leq C\,\|\partial_t(u-u_h)\|_{L^2(L^2)}\,.$$

□

Now, let us define the Ritz projection operator $R_h : H_0^1(\Omega) \to V_h$ by

$$A_h(R_h y, \chi_h) = A_h(y, \chi_h), \quad \forall \chi_h \in V_h.$$

With the help of the Ritz projection defined above and the analogous steps involved in the proof of Theorem 4.1 of [48], for a given $\tilde{u}$ one can easily obtain the following estimates.

Lemma 4.3.3. *Let us assume $g_h(x) = R_h g(x)$ and $w_h(x) = R_h w(x)$. Then there exists* *a positive constant C independent of h such that*

$$\|y(\tilde{u}) - y_h(\tilde{u})\|_{L^2(V(h))} + \|p(\tilde{y}) - p_h(\tilde{y})\|_{L^2(V(h))} + \|p(\tilde{u}) - p_h(\tilde{u})\|_{L^2(V(h))} \leq Ch, \tag{4.16}$$

$$\|y(\tilde{u}) - y_h(\tilde{u})\|_{L^2(L^2)} + \|p(\tilde{y}) - p_h(\tilde{y})\|_{L^2(L^2)} + \|p(\tilde{u}) - p_h(\tilde{u})\|_{L^2(L^2)} \leq Ch^2. \tag{4.17}$$

In particular, for $\tilde{u} = u_h$ we have

$$\|p(u_h) - p_h(u_h)\|_{L^2(L^2)} \leq Ch^2. \tag{4.18}$$

4.3.1 Error estimates for control

We will derive the estimates for $\|u - u_h\|_{L^2(L^2)}$ with three different control discretization approaches as mentioned earlier.

With variational approach:

In this approach the control space is not discretized explicitly and we have $U_{h,ad} = U_{ad}$.

Theorem 4.3.4. *Let u be a fixed local control of the problem (4.3) with associated state y and costate p and let (u_h, y_h, p_h) be their DFV approximations with variational discretization approach, then the following results hold true*

$$\|u - u_h\|_{L^2(L^2)} \leq Ch^2.$$

Proof. Employing the continuous (4.4) and discrete (4.8) variational inequalities with variational discretization approach, it is easy to obtain

$$(\lambda u_h + \mathcal{B}^* p_h, u - u_h)_{L^2(L^2)} \geq 0 \geq (\lambda u + \mathcal{B}^* p, u - u_h)_{L^2(L^2)}. \tag{4.19}$$

Using the coercivity (4.5) of j'', above relation (4.19), estimate (4.18) and adapting the similar arguments used in Theorem 3.3.5, we can obtain the desired estimate for control error. □

With piecewise constant discretization:

Next, we will establish the error estimate for the control variable u in the L^2-norm, i.e., $\|u - u_h\|_{L^2(L^2)}$ when the control is discretized by piecewise constant polynomials in space.

Theorem 4.3.5. *Let u be a local optimal control of the problem (4.3) and u_h be the solution of the discrete problem (4.6)-(4.8) with piecewise constant control discretization technique, then the following convergence result holds.*

$$\|u - u_h\|_{L^2(L^2)} \leq Ch.$$

Proof. In order to establish the proof, we follow the analysis in Theorem 3.3.6 and again utilize the coercivity of j''. We start by introducing an L^2-projection operator $\Pi_0 : U \longrightarrow U_h$ with the following approximation property: There exists a positive constant C independent of h such that

$$\|\tilde{u} - \Pi_0 \tilde{u}\|_{0,K} \leq Ch \|\tilde{u}\|_{1,K}.$$

Using the relation

$$(\lambda u_h + \mathcal{B}^* p_h, \Pi_0 u - u_h)_{L^2(L^2)} \geq 0 \geq (\lambda u + \mathcal{B}^* p, u - u_h)_{L^2(L^2)},$$

applying second order sufficient condition (3.8) for $u - u_h \in U$, estimate (4.18), continuity of operator $\mathcal{B}$, orthogonal and approximation property of projection Π_0, we can obtain

$$\|u - u_h\|^2_{L^2(L^2)} \leq Ch^2 \|u - u_h\|_{L^2(L^2)} + Ch^2 \|p_h\|_{L^2(V(h))} \|u\|_{L^2(H^1)}. \tag{4.20}$$

Testing the discrete state equation (4.6) for $v_h = \partial_t y_h$, employing the coercivity of $A_h(\cdot, \cdot)$, estimate (2.18) of Lemma 2.2.2, properties of nonlinear term and applying Gronwall's inequality, we can obtain the bound

$$\|y_h\|_{L^2(V(h))} \leq C \left(\|u_h\|_{L^2(L^2)} + \|f\|_{L^2(L^2)} \right). \tag{4.21}$$

Similarly, from the discrete costate equation, we can obtain

$$\|p_h\|_{L^2(V(h))} \leq C\left(\|y_h\|_{L^2(L^2)} + \|y_d\|_{L^2(L^2)}\right). \tag{4.22}$$

The uniform boundedness of p_h can be achieved from (4.21) and (4.22) by utilizing property (2.17) and the fact that $U_{h,ad}$ is uniformly bounded. The proof follows by applying Young's inequality in (4.20). □

With piecewise linear discretization:

In order to derive the error estimates $\|u - u_h\|_{L^2(L^2)}$ with piecewise linear control discretization approach, we make similar assumptions on on the structure of the active sets as assumed for semilinear parabolic case.

Theorem 4.3.6. *Let u be a local optimal control of the problem* (4.3) *and u_h be the solution of the discrete problem* (4.6)-(4.8) *with piecewise linear control discretization technique, then we have the following convergence result.*

$$\|u - u_h\|_{L^2(L^2)} \leq Ch^{3/2}.$$

Proof. To this end, we would like to mention that the proof of Theorem 4.3.6 is analogous to the proof of Theorem 3.3.9. However, for the sake of completeness, we only give main ideas of the proof. Here, again the key idea is to make use of the coercivity of j'' and Lemma 3.3.8. From the continuous and discrete variational inequality, we have the relation

$$(\lambda u_h + \mathcal{B}^* p_h, \tilde{u}_h - u_h)_{L^2(L^2)} \geq 0 \geq (\lambda u + \mathcal{B}^* p, u - u_h)_{L^2(L^2)}. \tag{4.23}$$

On application of condition (3.8) for $u - u_h \in U$, we get

$$C\|u - u_h\|^2_{L^2(L^2)} \leq (\lambda u + \mathcal{B}^* p, u - u_h)_{L^2(L^2)} - (\lambda u_h + \mathcal{B}^* p(u_h), u - u_h)_{L^2(L^2)}.$$

Now using the result (4.23) in the above relation we can obtain

$$C\|u-u_h\|^2_{L^2(L^2)} \leq (\mathcal{B}^*(p_h-p(u_h)),u-u_h)_{L^2(L^2)}+(\lambda(u-u_h)+\mathcal{B}^*(p-p_h),u-\tilde{u}_h)_{L^2(L^2)}$$
$$+(\lambda u+\mathcal{B}^*p,\tilde{u}_h-u)_{L^2(L^2)},$$

and therefore, by applying Cauchy-Schwarz inequality, using (4.18), property (2.17) and results of Lemma 4.3.1 and Theorem 4.17, we get

$$\|u-u_h\|^2_{L^2(L^2)} \leq Ch^2\|u-u_h\|_{L^2(L^2)}+\|u-u_h\|_{L^2(L^2)}\|u-\tilde{u}_h\|_{L^2(L^2)} \quad (4.24)$$
$$+|(\alpha u+B^*p,\tilde{u}_h-u)_{L^2(L^2)}|.$$

We use the definition 3.31 on the sets $\mathcal{T}_h^1$, $\mathcal{T}_h^2$ and $\mathcal{T}_h^3$ alongwith the projection property (3.7) and assumption 3.3.7, to get

$$\|u-\tilde{u}_h\|_{L^2(L^2)} \leq \frac{C}{\lambda}\left(h^2\left\|\nabla^2 p\right\|_{L^2(L^2)}+h^{3/2}\|\nabla p\|_{L^2(L^\infty)}\right). \quad (4.25)$$

Using the above estimate (4.25) in (4.24), applying Young's inequality and Lemma 3.3.8 we can obtain $\|u-u_h\|_{L^2(L^2)} \leq Ch^{3/2}$. □

4.3.2 Error estimates for state and costate

With variational discretization of control:

With variational discretization approach we can directly obtain the optimal estimates

$$\|y-y_h\|_{L^2(L^2)}+\|p-p_h\|_{L^2(L^2)} \leq Ch^2.$$

by decomposing the error for state as $y-y_h=y-y_h(u)+y_h(u)-y_h$ and for costate as $p-p_h=p-p_h(y)+p_h(y)-p_h$, applying triangle inequality together with the results of Lemma 4.3.1, (4.17) and the estimate $\|u-u_h\|_{L^2(L^2)} \leq Ch^2$.

With explicit control discretization:

As seen before for variational discretization approach one can obtain optimal convergence order for state and costate error without much difficulty. But if we follow analogously with the piecewise constant or linear control discretization techniques then we end up with suboptimal order of convergence. Using a more detailed analysis we can overcome this difficulty and obtain optimal convergence of $\mathcal{O}(h^2)$ for these two different schemes. For both choices of the space U_h (piecewise constant and linear) as described in Section 2.2.2 the following results hold.

Theorem 4.3.7. *Let u be a fixed local control of the problem (4.3) with associated state y and costate p and let (u_h, y_h, p_h) be their DFV approximations, then we have*

$$\|y - y_h\|_{L^2(L^2)} + \|p - p_h\|_{L^2(L^2)} \leq Ch^2.$$

Proof. Analogous to semilinear parabolic case, here also we start by decomposing $y - y_h = (y - y_h(u)) + (y_h(u) - y_h(\Pi_h u)) + (y_h(\Pi_h u) - y_h)$ and apply triangle inequality to obtain the relation

$$\begin{aligned} \|y - y_h\|_{L^2(L^2)} \leq \|y - y_h(u)\|_{L^2(L^2)} &+ \|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)} \\ &+ \|y_h(\Pi_h u) - y_h\|_{L^2(L^2)}, \end{aligned} \tag{4.26}$$

where, Π_h is the L^2 projection operator onto U_h. We assume $\tilde{p}_h(t,\cdot) \in V_h$, $(0 < t \leq T)$ to be the solution of auxiliary discrete dual equation

$$\begin{aligned} (\xi, \partial_{tt}\tilde{p}_h) + a_h(\tilde{p}_h, \xi) = (\xi, y_h(u) - y_h(\Pi_h u)) - (\xi, \hat{\varphi}\tilde{p}_h), \quad \forall \xi \in V_h \\ \tilde{p}_h(T,x) = 0, \quad \partial_t\tilde{p}_h(T,x) = 0 \end{aligned} \tag{4.27}$$

with

$$\hat{\varphi}(t,x) = \begin{cases} \frac{\varphi(y_h(u)) - \varphi(y_h(\Pi_h u))}{y_h(u) - y_h(\Pi_h u)}, & \text{if} \quad y_h(u) \neq y_h(\Pi_h u)) \\ 0, & \text{else.} \end{cases}$$

Let us choose $\xi = \partial_t\tilde{p}_h$ in (4.27) then on using ellipticity of $a_h(\cdot,\cdot)$, Gronwall's inequality, we can obtain the relation

$$\|\tilde{p}_h\|_{L^\infty(V(h))} \leq C \|y_h(u) - y_h(\Pi_h u)\|_{L^2(L^2)}. \tag{4.28}$$

Choosing $\xi = y_h(u) - y_h(\Pi_h u)$ in (4.27), we have

$$(y_h(u) - y_h(\Pi_h u), \partial_{tt}\tilde{p}_h) + a_h(\tilde{p}_h, y_h(u) - y_h(\Pi_h u))$$
$$= \|y_h(u) - y_h(\Pi_h u))\|_{0,\Omega}^2 - (\varphi(y_h(u)) - \varphi(y_h(\Pi_h u)), \tilde{p}_h). \tag{4.29}$$

From the discrete state equation for $y_h(u)$ and $y_h(\Pi_h u)$, we get

$$(\partial_{tt}(y_h(u) - y_h(\Pi_h u)), \gamma\tilde{p}_h) + A_h(y_h(u) - y_h(\Pi_h u), \tilde{p}_h)$$
$$= (\mathcal{B}(u - \Pi_h u), \gamma\tilde{p}_h) - (\varphi(y_h(u)) - \varphi(y_h(\Pi_h u)), \gamma\tilde{p}_h). \tag{4.30}$$

On subtracting (4.30) from (4.29), integrating from 0 to T and rearranging the terms , we an obtain

$$\|y_h(u) - y_h(\Pi_h u))\|_{L^2(L^2)}^2$$
$$= \int_0^T (\mathcal{B}(u - \Pi_h u), \gamma\tilde{p}_h) d\tau + \int_0^T \epsilon_a(y_h(u) - y_h(\Pi_h u), \tilde{p}_h) d\tau$$
$$+ \int_0^T (\varphi(y_h(u)) - \varphi(y_h(\Pi_h u)), \tilde{p}_h - \gamma\tilde{p}_h) d\tau + \int_0^T (\partial_{tt}(y_h(u) - y_h(\Pi_h u)), \tilde{p}_h - \gamma\tilde{p}_h) d\tau \tag{4.31}$$

Following similar steps as in the proof of Lemma 4.3.1 and Lemma 4.3.2, it is easy to establish

$$\|y_h(u) - y_h(\Pi_h u)\|_{L^\infty(V(h))} \leq \|u - \Pi_h u\|_{L^2(L^2)}, \tag{4.32}$$
$$\|\partial_{tt}(y_h(u) - y_h(\Pi_h u))\|_{L^\infty(L^2)} \leq \|\partial_t(u - \Pi_h u)\|_{L^2(L^2)}. \tag{4.33}$$

The first three terms of (4.31) can be bounded analogously as in the proof of Theorem 3.3.11. In order to bound the third term of 4.31, we proceed as follows. The approximation property of γ, above result (4.33) and result (4.28) readily gives

$$\int_0^T (\partial_{tt}(y_h(u) - y_h(\Pi_h u)), \tilde{p}_h - \gamma\tilde{p}_h) d\tau$$

$$\leq Ch \left\| \partial_{tt}(y_h(u) - y_h(\Pi_h u)) \right\|_{L^2(L^2))} \left\| \tilde{p}_h \right\|_{L^2(V(h))}$$

$$\leq Ch \left\| \partial_{tt}(y_h(u) - y_h(\Pi_h u)) \right\|_{L^\infty(L^2)} \left\| \tilde{p}_h \right\|_{L^\infty(V(h))}$$

$$\leq Ch \left\| \partial_t(u - \Pi_h u) \right\|_{L^2(L^2)} \left\| y_h(u) - y_h(\Pi_h u) \right\|_{L^2(L^2)}.$$

On substituting the bounds of the terms in (4.31), it is easy to obtain the estimate

$$\| y_h(u) - y_h(\Pi_h u) \|_{L^2(L^2)} \leq Ch \left(\| u - \Pi_h u \|_{L^2(L^2)} + \| \partial_t(u - \Pi_h u) \|_{L^2(L^2)} \right). \quad (4.34)$$

For the third term in (4.26), proceeding with similar arguments used in the proof of Theorem 3.3.11, we can obtain

$$\| y_h(\Pi_h u) - y_h \|_{L^2(L^2)} \leq Ch \left(\| u - \Pi_h u \|_{L^2(L^2)} + \| \partial_t(u - \Pi_h u) \|_{L^2(L^2)} \right). \quad (4.35)$$

Putting the bounds of (4.34) and (4.35) in (4.31), using the estimates of Theorem 4.17 and approximation properties of $\Pi_h u$, the optimal order of convergence for state with piecewise constant or piecewise linear discretization of control can be obtained, i.e.,

$$\| y - y_h \|_{L^2(L^2)} = \mathcal{O}(h^2). \quad (4.36)$$

On utilizing the estimates of Lemma 4.3.1, Theorem 4.17 and above result (4.36) it is easy to derive

$$\begin{aligned} \| p - p_h \|_{L^2(L^2)} &\leq \| p - p_h(y) \|_{L^2(L^2)} + \| p_h(y) - p_h \|_{L^\infty(V(h))} \\ &\leq \| p - p_h(y) \|_{L^2(L^2)} + \| y - y_h \|_{L^2(L^2)} = \mathcal{O}(h^2). \end{aligned}$$

□

In mesh dependent norm:

Theorem 4.3.8. *Let u be a fixed local control of the problem (4.3) with associated state y and costate p and let (u_h, y_h, p_h) be their DFV approximations, then we can obtain the following estimates in mesh-dependent norm*

$$\|y - y_h\|_{L^2(V(h))} + \|p - p_h\|_{L^2(V(h))} \leq Ch.$$

The above results can be readily obtained by using the standard result (4.16) and inserting the estimates of $\|u - u_h\|_{L^2(L^2)}$ and $\|y - y_h\|_{L^2(L^2)}$ in

$$\|y - y_h\|_{L^2(V(h))} \leq \|y - y_h(u)\|_{L^2(V(h))} + \|u - u_h\|_{L^2(L^2)} \text{ and}$$
$$\|p - p_h\|_{L^2(V(h))} \leq \|p - p_h(y)\|_{L^2(V(h))} + \|y - y_h\|_{L^2(L^2)}.$$

4.4 Fully discrete scheme

For the time discretization, we consider the partition $0 = t_0 < t_1 < ...t_M = T$ of the time interval $(0, T]$ with step size $k = \frac{T}{M}$. Let $\partial_{tt} y_h^i = \dfrac{y_h^{i+1} - 2y_h^i + y_h^{i-1}}{k^2}$ where $y_h^i = y_h(t_i, x)$, then we have the following scheme: Find $(y_h^i, p_h^i, u_h^i) \in V_h \times V_h \times U_{h,ad}$ such that for all $v_h, q_h \in V_h$

$$(\partial_{tt} y_h^i, \gamma v_h) + A_h(y_h^i, v_h) + (\varphi(y_h^i), \gamma v_h) = (\mathcal{B}u_h^i + f^i, \gamma v_h),\ i = 0, 1, .., M; \tag{4.37}$$
$$y_h^0(x) = g_h(x), \quad \partial_t y_h^0(x) = w_h(x), \quad x \in \Omega,$$
$$(\partial_{tt} p_h^i, \gamma q_h) + A_h(p_h^i, q_h) + (\varphi'(y_h^i) p_h^i, \gamma q_h) = (y_h^i - y_d^i, \gamma q_h),\ i = M, ..., 1, 0; \tag{4.38}$$
$$p_h^M(x) = 0, \quad \partial_t p_h^M(x) = 0, \quad x \in \Omega,$$
$$(\lambda u_h^i + \mathcal{B}^* p_h^i, \tilde{u}_h - u_h^i) \geq 0 \quad \forall \tilde{u}_h \in U_{h,ad}, \ \ i = 0, 1, ..., M. \tag{4.39}$$

We stress that convergence analysis of the above mentioned fully discrete scheme can be easily carried out in the similar fashion as we have derived for parabolic case. Therefore, we refrain ourself for providing the detailed proof, and directly state the following estimates.

Theorem 4.4.1. *Let u be a fixed local optimal control of problem (4.3) and u_h^m be the* *solution of the fully discrete optimal control problem* (4.37)-(4.39) *at* $t = t_m$*, then the following error estimate holds*

$$\|u - u_h\|_{L^2(L^2)} \le C(h^2 + k), \quad \textit{(with variational discretization approach)},$$
$$\|u - u_h\|_{L^2(L^2)} \le C(h^{3/2} + k), \quad \textit{(with piecewise linear discretization approach)},$$
$$\|u - u_h\|_{L^2(L^2)} \le C(h + k), \quad \textit{(with piecewise constant discretization approach)}.$$

Theorem 4.4.2. *Let* u *be an optimal control of problem* (4.3) *with the associated state* y *and costate* p*, respectively, and let* u_h^m*,* y_h^m *and* p_h^m *be the solution of the fully discrete optimal control problem* (4.37)-(4.39) *at* $t = t_m$*, then the following discretization error estimates are satisfied*

$$\|y - y_h\|_{L^2(L^2)} + \|p - p_h\|_{L^2(L^2)} \le C(h^2 + k),$$
$$\|y - y_h\|_{L^2(V(h))} + \|p - p_h\|_{L^2(V(h))} \le C(h + k).$$

4.5 Numerical Experiments

In this Section, we present two numerical examples to illustrate the performance of the proposed scheme applied to distributed semilinear hyperbolic optimal control problem.

Implementation aspects

Analogous to the semilinear parabolic case, for computational aspects here also we have used the idea of interpolated coefficients to approximate the nonlinear term. We recall that discontinuous interpolation operator $I_h : C(\Omega) \longrightarrow V_h$ is defined by

$$(I_h v)|_K := \sum_{i=1}^{3} v_i \phi_i, \quad K \in \mathcal{T}_h,$$

where $\{\phi_i\}_{i=1}^3$ be the standard local basis functions for the finite dimensional space V_h associated with triangle K and $v_i's$ are the nodal values of function v on triangle $K \in \mathcal{T}_h$. Employing the method of interpolated coefficients, the discrete equation

(4.37) for $i = 0, ..., M$ can be reformulated as

$$(\partial_{tt} y_h^i, \gamma v_h) + A_h(y_h^i, v_h) + (I_h\varphi(y_h^i), \gamma v_h) = (\mathcal{B}u_h^i + f^i, \gamma v_h),\ i = 0, 1, .., M;$$

which leads to a nonlinear system of equations

$$\mathbb{M}\frac{Y^{i+1} - 2Y^i + y^{i-1}}{k^2} + \mathbb{A}Y^{i+1} + \mathbb{M}\varphi(Y^{i+1}) = \mathbb{G}U^{i+1} + \mathbb{F}^{i+1}.$$

which we solve by Newton method. The Jacobian matrix is $\mathbb{J} = (\mathbb{M} + k^2\mathbb{A}) + k^2\mathbb{M}_{\varphi'}$.

In order to validate the theoretical error estimates derived for control, state and costate variables, we consider the following example.

Example 4.5.1.

$$\min_{u\in U_{ad}} J(y, u) := \frac{1}{2}\int_0^1 \|(y(t, x) - y_d(t, x))\|_{0,\Omega}^2 \, dt + \frac{1}{2}\int_0^1 \|u(t, x)\|_{0,\Omega}^2 \, dt,$$

subject to

$$\begin{aligned}
&\partial_{tt} y - \Delta y + y^3 = u + f, \quad \text{in} \quad (0, 1] \times \Omega,\\
&y(t, x) = 0, \quad \text{on} \quad (0, 1] \times \partial\Omega,\\
&y(0, x) = \sin(\pi x_1)\sin(\pi x_2),\ \partial_t y(0, x) = \sin(\pi x_1)\sin(\pi x_2),\ \text{in } \Omega.
\end{aligned}$$

Here, the space domain $\Omega = \{x = (x_1, x_2) : 0 \le x_1 \le 1, 0 \le x_2 \le 1\}$, the source term $f(t, x)$ and the desired state $y_d(t, x)$ are of the form

$$\begin{aligned}
f &= (1 + 2\pi^2)e^t \sin(\pi x_1)\sin(\pi x_2) + e^{3t}\sin(\pi x_1)^3\sin(\pi x_2)^3 - u(t, x),\\
y_d &= \sin(\pi x_1)\sin(\pi x_2)(e^t + 2 + 2\pi^2(t - 1)^2) + 3e^{2t}(t - 1)^2\sin(\pi x_1)^3\sin(\pi x_2)^3.
\end{aligned}$$

To assess the experimental convergence, we would require the exact solution of the above mentioned control problem. Therefore, with the choice of the source term f and the desired state y_d, the exact state y and the costate p is given in the following manner

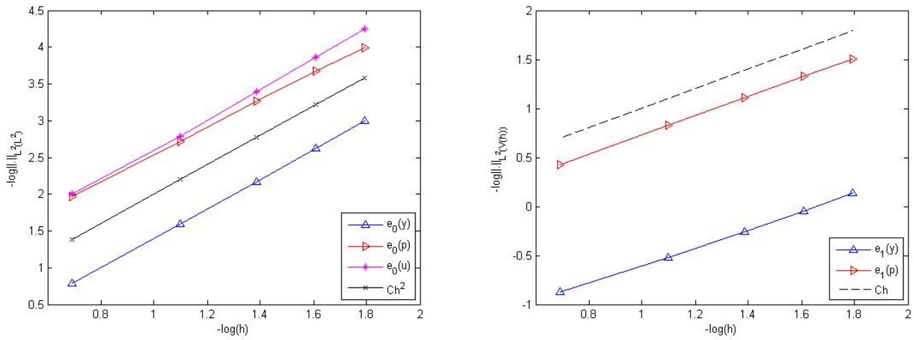
$$y(t, x) = e^t \sin(\pi x_1)\sin(\pi x_2), \quad p(t, x) = -(t - 1)^2\sin(\pi x_1)\sin(\pi x_2).$$

Moreover, the control variable is defined as: $u(t, x) = max(0, min(1, -p(t, x)))$.

h	$e_0(y)$	$r_0(y)$	$e_1(y)$	$r_1(y)$	$e_0(p)$	$r_0(p)$	$e_1(p)$	$r_1(p)$	$e_0(u)$	$r_0(u)$
Variational discretization approach										
0.5000	0.4575	-	2.3963	-	0.1394	-	0.6509	-	0.1342	-
0.3333	0.2057	1.9711	1.6889	0.8628	0.0659	1.8448	0.4376	0.9790	0.0612	1.9340
0.2500	0.1168	1.9656	1.2954	0.9218	0.0379	1.9229	0.3306	0.9747	0.0334	2.0989
0.2000	0.0751	1.9817	1.0462	0.9573	0.0251	1.8376	0.2655	0.9824	0.0208	2.1165
0.1666	0.0522	1.9866	0.8758	0.9753	0.0184	1.7108	0.2218	0.9853	0.0142	2.0985
Piecewise constant control										
0.5000	0.4599	-	2.3995	-	0.1375	-	0.6448	-	0.1472	-
0.3333	0.2087	1.9489	1.6907	0.8634	0.0646	1.8602	0.4354	0.9687	0.0926	1.1416
0.2500	0.1190	1.9526	1.2963	0.9231	0.0372	1.9226	0.3297	0.9666	0.0662	1.1650
0.2000	0.0766	1.9732	1.0467	0.9583	0.0247	1.8351	0.2650	0.9783	0.0516	1.1175
0.1666	0.0534	1.9804	0.8761	0.9760	0.0181	1.7066	0.2215	0.9829	0.0424	1.0777
Piecewise linear control										
0.5000	0.4593	-	2.3980	-	0.1380	-	0.6464	-	0.1168	-
0.3333	0.2076	1.9578	1.6897	0.8634	0.0651	1.8525	0.4361	0.9705	0.0598	1.6485
0.2500	0.1178	1.9684	1.2956	0.9231	0.0375	1.9110	0.3301	0.9677	0.0357	1.7875
0.2000	0.0756	1.9849	1.0463	0.9578	0.0249	1.8302	0.2653	0.9797	0.0241	1.7561
0.1666	0.0525	2.0008	0.8757	0.9758	0.0183	1.6944	0.2217	0.9831	0.0179	1.6383

Table 4.1: The development of the errors with spatial triangulation and fixed time step size $k = 0.01$ for state, costate and control variables.

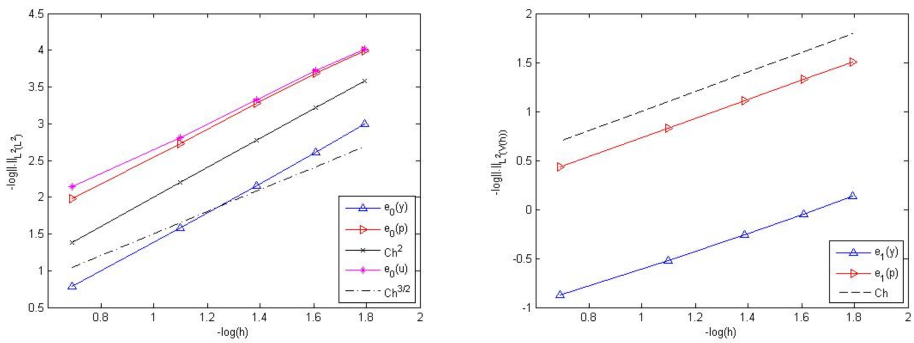
We will use the notations (3.69) and (3.70), defined in Section 3.5 of Chapter 3 to measure errors for optimal state, costate and control variables and corresponding observed rates.



(a) Convergence of state, costate and control. (b) Convergence of state and costate.

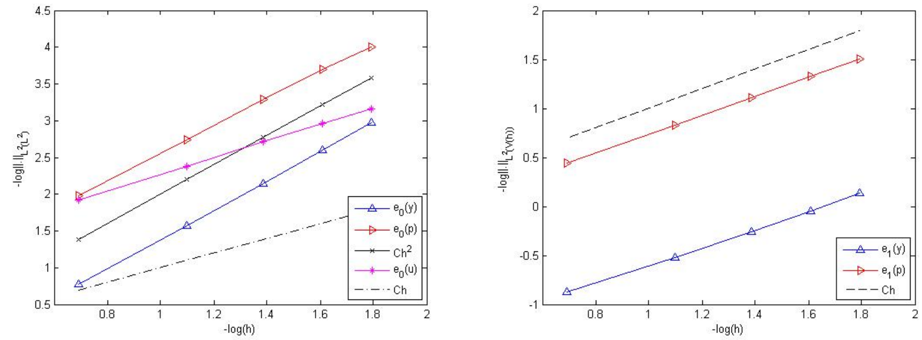
Figure 4.1: The order of convergence of the errors of DFV discretization of the state, costate and control variables with variational discretization approach computed with $\theta = -1$, $\beta = 1$, $\alpha = 10$ and time step size $k = 0.01$.

The results concerning the errors of approximation for the optimal state, costate and control variables for three different control discretization approach on a sequence of uniformly refined meshes with fixed time step length $k = 0.01$ are reported in Table 4.1. The corresponding convergence orders are shown in Figures 4.1, 4.2 and 4.3 which is in agreement with the theoretical results.



(a) Convergence of state, costate and control. (b) Convergence of state and costate.

Figure 4.2: The order of convergence of the errors of DFV discretization of the state, adjoint state and control variables with piecewise linear discretization of control for $\theta = -1$, $\beta = 1$, $\alpha = 10$ and time step size $k = 0.01$.



(a) Convergence of state, costate and control. (b) Convergence of state and costate.

Figure 4.3: The convergence order of the errors of DFV approximations of the state, costate and control variables using piecewise constant discretization of control which are computed for $\theta = -1$, $\beta = 1$, $\alpha = 10$ and time step $k = 0.01$.

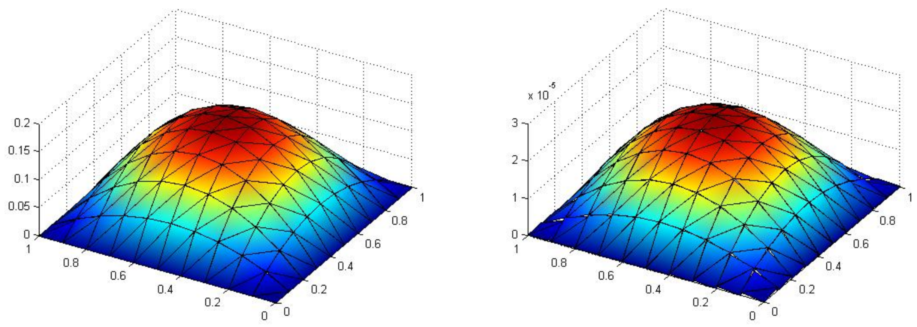
To illustrate the performance of the proposed numerical scheme we consider another example corresponding to (4.2)-(4.2) in which exact solutions are not available.

Example 4.5.2. The problem represents the optimal oscillations of a membrane which is fixed on the boundary. The domain consists of unit square and the final time $T = 1$. The displacement and velocity at time zero are given by the initial data $g(x) = w(x) = (x_1^2 - x_1)(x_2^2 - x_2)$. The applied body force f and the target function y_d are

$$\begin{aligned} f &= e^t[(x_1^2 - x_1)(x_2^2 - x_2) - 2(x_1^2 - x_1 + x_2^2 - x_2)] + sin(e^t(x_1^2 - x_1)(x_2^2 - x_2)) \\ &\quad -max(0, min(0.8, 2(t-1)^2(x_1^2 - x_1)(x_2^2 - x_2))) \quad \text{and} \\ y_d &= (e^t + 2)(x_1^2 - x_1)(x_2^2 - x_2) + (t-1)^2[(x_1^2 - x_1)(x_2^2 - x_2) \\ &\quad cos(e^t(x_1^2 - x_1)(x_2^2 - x_2)) - 2(x_1^2 - x_1 + x_2^2 - x_2)], \text{ respectively.} \end{aligned}$$

The nonlinear term is $\varphi(y) = sin(y)$, the control bounds are $u_a = 0$, $u_b = 0.8$ and the regularization parameter is $\lambda = 0.5$.

The computed optimal control acting as a force on the membrane and the corresponding displacement at final time T with mesh size $h = 0.1$ and time step length $k = 0.01$ are depicted in Figure 4.4. In addition, the effect of control cost on the the minimum values of objective functional is listed in Table 4.2.



(a) The computed optimal state. (b) The computed optimal control.

Figure 4.4: The DFV approximation of optimal control and associated state with piecewise linear discretization of control with $\theta = -1$, $\beta = 1$ and $\alpha = 10$.

λ	0.1	0.01	0.001	0.0001	0.00001
$J(y_h, u_h)$	0.0709148	0.0703449	0.0682066	0.0677848	0.0677252

Table 4.2: The values of objective functional for different regularization parameter for the DFV approximations of the semilinear hyperbolic optimal control problem.

Remark 4.5.3. *For our numerical experiments, we have considered $\theta = -1$ (SIPG). However, we have observed similar rate of convergence for the other two cases $\theta = 1$ (NIPG) and $\theta = 0$ (IIPG).*

CHAPTER 5

Optimal control problem governed by Brinkman equations

In this Chapter, we describe discontinuous finite volume approximations for optimal control problems governed by the Brinkman equations written in terms of velocity and pressure. An additional force field is sought that produces a velocity close to a desired known value. The discretization of state and costate velocity and pressure follows a lowest order DFV scheme, whereas three different approaches are used for the control approximation: variational discretization, element-wise constant and element-wise linear functions. Here also we have employed *optimize-then-discretize* approach, and the resulting discrete formulation is nonsymmetric. We derive *a priori* error estimates for velocity, pressure and control in natural norms. A set of numerical examples is finally presented to illustrate the performance of the method and to confirm the predicted accuracy of the state, costate and control approximations under various scenarios including 2D and 3D cases.

5.1 Introduction

Fluid control problems are highly important in the field of science and engineering. They are often useful to minimize drag, to increase mixing properties, to reduce turbulent kinetic energy, and several other features.

Theoretical aspects of these control problems can be found in the classical works [1, 54]. Regarding their numerical solution, the literature is abundant, especially if associated to FE methods (see e.g.,[11, 34, 38, 69, 74, 78] and the references therein). Most contributions in the context of Stokes and Navier-Stokes approximation employ conforming discretizations for state, costate and control variables. In this case, it has been found that the convergence rate of the control approximation is of $\mathcal{O}(h)$ and $\mathcal{O}(h^{\frac{3}{2}})$ for piecewise constant and piecewise linear discretizations, respectively. On the other hand, using the so-called variational discretization approach (cf. [42], in which the control set is not discretized explicitly but recovered by a projection), an improved convergence of $\mathcal{O}(h^2)$ was obtained. A similar result holds if using graded meshes instead of uniform partitions [68]. A few results are also available for finite differences [33], spectral [22], mimetic [3], fully-mixed [7], and DG [18, 19, 23] methods applied to flow control problems. On the other hand, motivated by local conservation properties, a priori error estimates of FVE approximations of linear elliptic and parabolic optimal control problems have been established in [59, 60], employing a variational discretization approach.

We recall that two main strategies are available for the numerical solution of optimal control problems: the so-called *optimize-then-discretize* approach and *discretize-then-optimize*. It is well-known that for non-symmetric discrete formulations, these two approaches may lead to different discrete adjoint equations and the solutions may not coincide. In general, finite volume (FV) and related schemes are not necessarily symmetric and a choice of the appropriate strategy should be based on both theoretical and computational considerations. For instance, FVE methods were employed in [59, 60] together with *optimize-then-discretize* approach for the approximation of elliptic and parabolic optimal control problems. Here we will adopt *optimize-then-discretize* strategy.

In contrast with the condensed review given above, here we will focus on DFV methods for the approximation of optimal control problems. We also recall the fact that in DFV methods, discontinuous piecewise linear functions conform the trial space, whereas piecewise constant test functions are used in a finite volume fashion. The application of DFV methods in the approximation of Stokes and related fluid problems can be found in e.g. [12, 32, 50, 52, 82]. In this Chapter our objective is to apply DFV schemes to the case of velocity control for the linear Brinkman equations. For the approximation of the control variable, we will discuss three alternatives: a variational discretization approach, element-wise constant and element-wise linear discretization.

Notations: Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded convex polygonal domain with boundary $\partial\Omega$. The outward unit normal vector to Ω is denoted by $\boldsymbol{n}$. Standard terminology will be employed for Sobolev spaces: $\mathbf{H}^1(\Omega) = H^1(\Omega)^d$ and $\mathbf{H}^1_0(\Omega) := \{\boldsymbol{v} \in \mathbf{H}^1(\Omega) : \boldsymbol{v}|_{\partial\Omega} = \mathbf{0}\}$. The corresponding norms will be denoted by $\|\cdot\|_{1,\Omega}$. We also consider the space of integrable functions with zero mean: $L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q \,\mathrm{d}\boldsymbol{x} = 0\}$ and

$\mathbf{L}^2(\Omega) = L^2(\Omega)^d$. The notation $(\cdot,\cdot)_{0,\Omega}$ stands for the scalar product in $\mathbf{L}^2(\Omega)$ and we use $\|\cdot\|_{0,\Omega}$ to denote the associated norm. These notations will be frequently used along the Chapter. By $\mathbf{div}$ we will denote the usual divergence operator div applied row-wise to a tensor, $\mathbf{I}$ stands for $d \times d$ identity matrix, and $\mathbf{0}$ will be used as a generic null vector.

5.1.1 The Brinkman model problem

Optimal control problems governed by Brinkman equations describe the controlled motion of an incompressible viscous fluid within an array of porous particles. We investigate the following optimization problem with control variable $\boldsymbol{u}$, fluid velocity $\boldsymbol{y}$ and pressure field p:

$$\min_{\boldsymbol{u}\in\mathbf{U}_{\mathrm{ad}}} J(\boldsymbol{u}) := \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{y}_d\|^2_{0,\Omega} + \frac{\lambda}{2}\|\boldsymbol{u}\|^2_{0,\Omega}, \tag{5.1}$$

governed by the Brinkman equations

$$\left.\begin{aligned} \mathbf{K}^{-1}(\boldsymbol{x})\boldsymbol{y} - \mathbf{div}\,(\mu(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{y}) - p\mathbf{I}) = \boldsymbol{u} + \boldsymbol{f}, &\text{ in } \Omega,\\ \operatorname{div}\boldsymbol{y} = 0, \quad &\text{in } \Omega,\\ \boldsymbol{y} = \mathbf{0}, \quad &\text{on } \partial\Omega, \end{aligned}\right\} \tag{5.2}$$

The set of feasible controls $\mathbf{U}_{\mathrm{ad}}$ is defined by

$$\mathbf{U}_{\mathrm{ad}} = \{\boldsymbol{u} \in \mathbf{L}^2(\Omega) : u_{a_j} \leq u_j \leq u_{b_j} \text{ a.e. in } \Omega\}.$$

for $-\infty \leq u_{a_j} < u_{b_j} \leq \infty,\ j = 1,\dots,d$. The quantity $\mu(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{y}) - p\mathbf{I}$ is the Cauchy (true stress) tensor, where $\boldsymbol{\varepsilon}(\boldsymbol{y}) = \frac{1}{2}(\nabla\boldsymbol{y} + \nabla\boldsymbol{y}^T)$ is the infinitesimal rate of strain, $\mu(\boldsymbol{x})$ is the dynamic viscosity of the fluid, and $\mathbf{K}(\boldsymbol{x})$ stands for the permeability tensor of the medium (typically rescaled by the viscosity). As before, $\lambda > 0$ denotes a given Tikhonov regularization (or control cost) parameter. The desired velocity $\boldsymbol{y}_d$ and the applied body force $\boldsymbol{f}$ are known data with regularity $\mathbf{L}^2(\Omega)$ or $\mathbf{H}^1(\Omega)$, depending on the specific case. The whole idea of this problem is to identify an additional force $\boldsymbol{u}$ giving rise to a velocity $\boldsymbol{y}$ close to a known desired or target velocity $\boldsymbol{y}_d$.

We assume that $\mathbf{K}$ is symmetric, uniformly bounded and positive definite; and that viscosity and permeability satisfy

$$\begin{aligned}&\exists\,\gamma_1,\mu_{\min},\mu_{\max}>0\,:\,\forall s\in\mathbb{R}_+\,;\,\mu_{\min}<\mu(s)<\mu_{\max},\ |\mu'(s)|\leq\gamma_1,\\&\exists\,k_1,k_2>0\,:\,0<k_1\leq\mathbf{K}^{-1}(\boldsymbol{x})\leq k_2\quad\forall\boldsymbol{x}\in\Omega,\end{aligned}\tag{5.3}$$

the last inequalities being understood component-wise. The standard weak formulation of the state equations (5.2) is given by: find $(\boldsymbol{y},p)\in\mathbf{H}^1_0(\Omega)\times L^2_0(\Omega)$ such that

$$\left.\begin{aligned}a(\boldsymbol{y},\boldsymbol{v})+c(\boldsymbol{y},\boldsymbol{v})+b(\boldsymbol{v},p)=(\boldsymbol{f}+\boldsymbol{u},\boldsymbol{v})_{0,\Omega}\qquad&\forall\boldsymbol{v}\in\mathbf{H}^1_0(\Omega),\\ b(\boldsymbol{y},q)=0\qquad&\forall q\in L^2_0(\Omega),\end{aligned}\right\}\tag{5.4}$$

where the bilinear forms $a(\cdot,\cdot):\mathbf{H}^1_0(\Omega)\times\mathbf{H}^1_0(\Omega)\to\mathbb{R},\ c(\cdot,\cdot):\mathbf{H}^1_0(\Omega)\times\mathbf{H}^1_0(\Omega)\to\mathbb{R}$ and $b(\cdot,\cdot):\mathbf{H}^1_0(\Omega)\times L^2_0(\Omega)\to\mathbb{R}$ are defined as:

$$\begin{aligned}&a(\boldsymbol{y},\boldsymbol{v}):=\int_\Omega\mathbf{K}^{-1}(\boldsymbol{x})\boldsymbol{y}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x},\qquad c(\boldsymbol{y},\boldsymbol{v}):=\int_\Omega\mu(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{y}):\boldsymbol{\varepsilon}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{x},\\&b(\boldsymbol{v},q):=-\int_\Omega q\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}\boldsymbol{x},\end{aligned}$$

for all $\boldsymbol{y},\boldsymbol{v}\in\mathbf{H}^1_0(\Omega)$ and $q\in L^2_0(\Omega)$. Problem (5.4) satisfies the following Babuška-Brezzi condition (see [71], for example): there exists $\xi>0$ such that

$$\inf_{q\in L^2_0(\Omega)}\sup_{\mathbf{0}\neq\boldsymbol{v}\in\mathbf{H}^1_0(\Omega)}\frac{b(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{1,\Omega}\,\|q\|_{0,\Omega}}\geq\xi,$$

and its unique solvability is ensured [71].

The optimality condition can be formulated as

$$J'(\boldsymbol{u})(\tilde{\boldsymbol{u}}-\boldsymbol{u})\geq0,\quad\forall\tilde{\boldsymbol{u}}\in\mathbf{U}_{\mathrm{ad}},$$

which can be rewritten in the form:

$$(\boldsymbol{w}+\lambda\boldsymbol{u},\tilde{\boldsymbol{u}}-\boldsymbol{u})_{0,\Omega}\geq0\quad\forall\tilde{\boldsymbol{u}}\in\mathbf{U}_{\mathrm{ad}},\tag{5.5}$$

where $\boldsymbol{w}$ is the velocity associated to the adjoint equation

$$\left.\begin{aligned} \mathbf{K}^{-1}(\boldsymbol{x})\boldsymbol{w} - \mathbf{div}(\mu(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{w}) + r\mathbf{I}) = \boldsymbol{y} - \boldsymbol{y}_d \quad \text{in } \Omega, \\ \operatorname{div} \boldsymbol{w} = 0 \quad \text{in } \Omega, \\ \boldsymbol{w} = \mathbf{0} \quad \text{on } \partial\Omega. \end{aligned}\right\}$$

The variational inequality (5.5) can be equivalently recast in component-wise manner

$$u_j(\boldsymbol{x}) = P_{[u_{a_j}, u_{b_j}]}\left(\frac{-1}{\lambda} w_j(\boldsymbol{x})\right) \quad \text{a.e. in } \Omega, j = 1, \ldots, d,$$

where the operator $P_{[u_{a_j}, u_{b_j}]}$ denotes a projection defined for a generic scalar function z as

$$P_{[u_{a_j}, u_{b_j}]}(z(\boldsymbol{x})) = \max(u_{a_j}, \min(u_{b_j}, z(\boldsymbol{x}))), \quad \text{a.e. in } \Omega, j = 1, \ldots, d,$$

and if $z \in W^{1,\infty}(\Omega)$, it further satisfies

$$\left\|\nabla P_{[a_j,b_j]}(z)\right\|_{L^\infty(\Omega)} \leq \|\nabla z\|_{L^\infty(\Omega)}. \tag{5.6}$$

The remainder of this Chapter is structured in the following manner. In Section 5.2 we formulate the DFV scheme of the considered optimal control problem. Section 5.3 focuses on the development of *a priori* error estimates for different types of control discretizations. Finally, in Section 5.4 we summarize the solution algorithm and illustrate our theoretical error bounds and performance of the method by a set of numerical experiments.

5.2 Discretization

We first recall the construction of control volumes in DFV scheme as presented before in Section 2.2 of Chapter 2.

5.2.1 Meshes, discrete spaces, and interpolation properties

Let $\mathcal{T}_h$ be a regular, quasi-uniform partition of $\bar{\Omega} \subset \mathbb{R}^d$, $d = 2, 3$, into closed triangles (or tetrahedra if $d = 3$). By h_T we denote the diameter of a given element $T \in \mathcal{T}_h$, and the global meshsize by $h = \max_{T \in \mathcal{T}_h} h_T$. Moreover, let $\mathcal{E}_h$ and $\mathcal{E}_h^\Gamma$ denote, respectively, the set of all faces and boundary faces in $\mathcal{T}_h$ (edges and boundary edges if $d = 2$), and the symbol h_e represents the length of the edge e (or the area of the face e if $d = 3$). It follows from the definitions of h_e, h_T and h that

$$h_e \leq h_T^{d-1} \leq h^{d-1}. \tag{5.7}$$

In addition to $\mathcal{T}_h$, we introduce a dual partition in the following way. Each element $T \in \mathcal{T}_h$ is split into three sub triangles (or four sub-tetrahedra if $d = 3$) T_i^*, $i = 1, \dots, d+1$, by connecting the barycenter of the element to its corner nodes (see a schematic for $d = 2$ and $d = 3$ in Figure 5.1). The set of all these elements generated by barycentric subdivison will be denoted by $\mathcal{T}_h^*$ and will be called the *dual partition* of $\mathcal{T}_h$.

We recall the definition of jump and average defined in Chapter 2. Let e be an interior face shared by two elements T_1 and T_2 in $\mathcal{T}_h$. By $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ we will denote unit normal vectors on e pointing outwards T_1 and T_2, respectively. Then the average $\langle \cdot \rangle$ and jump $[\![\cdot]\!]$ operators defined on e for generic scalar and vector fields $q, \boldsymbol{v}$, respectively, are:

$$\langle q \rangle = \frac{1}{2}(q|_{\partial T_1} + q|_{\partial T_2}), \quad [\![q]\!] = q|_{\partial T_1} - q|_{\partial T_2},$$
$$\langle \boldsymbol{v} \rangle = \frac{1}{2}(\boldsymbol{v}|_{\partial T_1} + \boldsymbol{v}|_{\partial T_2}), \quad [\![\boldsymbol{v}]\!] = \boldsymbol{v}|_{\partial T_1} - \boldsymbol{v}|_{\partial T_2}.$$

If $e \in \mathcal{E}_h^\Gamma$, then we simply take $\langle q \rangle = q$ and $[\![\boldsymbol{v}]\!] = \boldsymbol{v}$. Notice that jump and averages are defined so that they preserve the dimension of the argument.

As usual, we denote by $\mathcal{P}_m(T)$ the space of polynomials of degree less or equal than m, defined on the element T, and $\mathbf{P}_m(T)$ will denote its vectorial counterpart. Then, a finite dimensional trial space (that will be used for the state and costate velocity approximation) associated with the primal partition $\mathcal{T}_h$ is given by

$$\mathbf{V}_h = \{\boldsymbol{v}_h \in \mathbf{L}^2(\Omega) : \boldsymbol{v}_h|_T \in \mathbf{P}_1(T), \ \forall T \in \mathcal{T}_h\}.$$

The finite dimensional test space for velocities and corresponding to the dual partition $\mathcal{T}_h^*$ is

$$\mathbf{V}_h^* = \{\boldsymbol{v}_h \in \mathbf{L}^2(\Omega) : \boldsymbol{v}_h|_{T^*} \in \mathbf{P}_0(T^*), \ \forall T^* \in \mathcal{T}_h^*\},$$

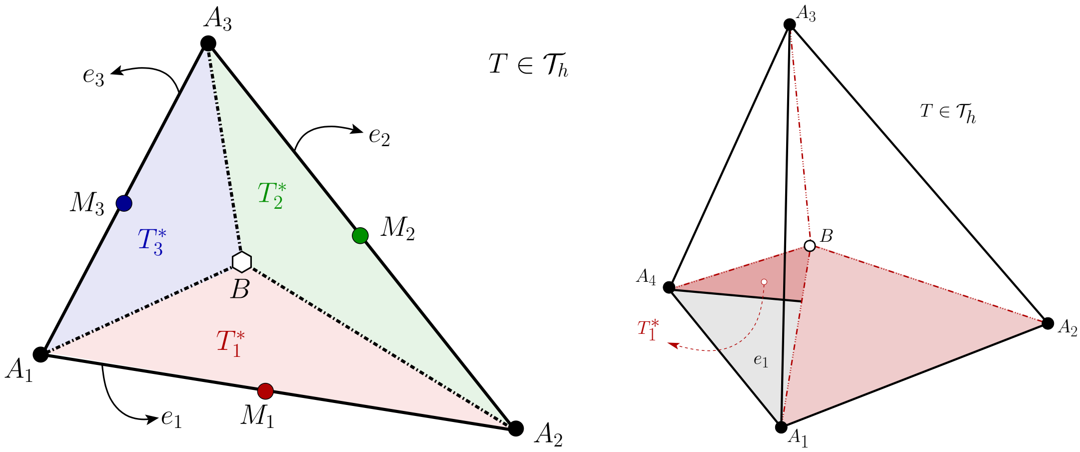


Figure 5.1: Left: sketch of a single primal element T in $\mathcal{T}_h$, and sub-elements T_i^* belonging to the dual partition $\mathcal{T}_h^*$. Right: its three-dimensional counterpart.

and the discrete space for state and costate pressure approximation is defined as

$$Q_h = \{q_h \in L_0^2(\Omega) : q_h|_T \in \mathcal{P}_0(T),\ \forall T \in \mathcal{T}_h\}.$$

In addition, we define a space with higher regularity

$$\mathbf{V}(h) = \mathbf{V}_h + [\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)],$$

and the connection between spaces associated to the two different meshes is characterized by the transfer operator $\gamma : \mathbf{V}(h) \to \mathbf{V}_h^*$, defined in the following manner:

$$\gamma \boldsymbol{v}|_{T^*} = \frac{1}{h_e} \int_e \boldsymbol{v}|_{T^*} \, \mathrm{d}s, \quad \text{for } T^* \in \mathcal{T}_h^*. \tag{5.8}$$

Some useful properties of this map are collected in the following result.

Lemma 5.2.1. *Let γ be a transfer operator defined as in (5.8). Then*

i) γ is self-adjoint with respect to the $\mathbf{L}^2$-inner product, i.e.

$$(\boldsymbol{v}_h, \gamma \boldsymbol{z}_h)_{0,\Omega} = (\boldsymbol{z}_h, \gamma \boldsymbol{v}_h)_{0,\Omega}, \quad \forall \boldsymbol{v}_h, \boldsymbol{z}_h \in \mathbf{V}_h.$$

ii) If $\|\boldsymbol{v}_h\|_{0,h}^2 := (\boldsymbol{v}_h, \gamma \boldsymbol{v}_h)_{0,\Omega}$, then $\|\cdot\|_{0,h}$ and $\|\cdot\|_{0,\Omega}$ are equivalent, with equivalence constants being independent of h.

iii) The operator γ is stable with respect to the norm $\|\cdot\|_{0,\Omega}$, that is

$$\|\gamma \boldsymbol{v}_h\|_{0,\Omega} = \|\boldsymbol{v}_h\|_{0,\Omega} \quad \forall \boldsymbol{v}_h \in \mathbf{V}_h. \tag{5.9}$$

iv) For all $\boldsymbol{v} \in \mathbf{V}(h)$ and $T \in \mathcal{T}_h$, we have

$$\int_e (\boldsymbol{v} - \gamma \boldsymbol{v}) \,\mathrm{d}s = \mathbf{0}, \qquad \int_e [\![\boldsymbol{v} - \gamma \boldsymbol{v}]\!]_e \,\mathrm{d}s = \mathbf{0},$$
$$\int_T (\boldsymbol{v} - \gamma \boldsymbol{v}) \,\mathrm{d}\boldsymbol{x} = \mathbf{0}, \qquad \|\boldsymbol{v} - \gamma \boldsymbol{v}\|_{0,T} \leq C h_T \|\boldsymbol{v}\|_{1,T}.$$

Proof. Different proofs can be found in e.g. [9, 10, 49]. □

Let $\boldsymbol{v}_h \in \mathbf{V}_h$. We proceed to test the state equation (5.2) against $\gamma \boldsymbol{v}_h \in \mathbf{V}_h^*$ and $\phi_h \in Q_h$, and after integrating by parts the momentum equation on each dual element and the mass equation on each primal element we end up with the following DFV scheme: Find $(\boldsymbol{y}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\hat{A}_h(\boldsymbol{y}_h, \boldsymbol{v}_h) + c_h(\boldsymbol{y}_h, \boldsymbol{v}_h) + C_h(\boldsymbol{v}_h, p_h) = (\boldsymbol{u}_h + \boldsymbol{f}, \gamma \boldsymbol{v}_h)_{0,\Omega} \quad \forall \boldsymbol{v}_h \in \mathbf{V}_h, \tag{5.10}$$
$$B_h(\boldsymbol{y}_h, \phi_h) = 0, \quad \forall \phi_h \in Q_h, \tag{5.11}$$

where the discrete bilinear forms $\hat{A}_h(\cdot,\cdot)$, $c_h(\cdot,\cdot)$, $C_h(\cdot,\cdot)$ and $B_h(\cdot,\cdot)$ are defined in the following manner (see also [12]):

$$\begin{aligned}
\hat{A}_h(\boldsymbol{w}_h, \boldsymbol{v}_h) &:= (\mathbf{K}^{-1}(\boldsymbol{x})\boldsymbol{w}_h, \gamma \boldsymbol{v}_h)_{0,\Omega}, \\
c_h(\boldsymbol{w}_h, \boldsymbol{v}_h) &:= -\sum_{T\in\mathcal{T}_h} \sum_{j=1}^{d+1} \int_{A_{j+1}BA_j} \mu(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{w}_h)\boldsymbol{n} \cdot \gamma \boldsymbol{v}_h \,\mathrm{d}s - \sum_{e\in\mathcal{E}_h} \int_e \langle \mu(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{w}_h)\boldsymbol{n} \rangle \\
&\quad \cdot [\![\gamma \boldsymbol{v}_h]\!] \,\mathrm{d}s - \sum_{e\in\mathcal{E}_h} \int_e \langle \mu(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\boldsymbol{n} \rangle \cdot [\![\gamma \boldsymbol{w}_h]\!] \,\mathrm{d}s + \sum_{e\in\mathcal{E}_h} \int_e \frac{\alpha_d}{h_e^\delta} [\![\boldsymbol{w}_h]\!] \cdot [\![\boldsymbol{v}_h]\!] \,\mathrm{d}s, \\
C_h(\boldsymbol{v}_h, q_h) &:= \sum_{T\in\mathcal{T}_h} \sum_{j=1}^{d+1} \int_{A_{j+1}BA_j} q_h \boldsymbol{n} \cdot \gamma \boldsymbol{v}_h \,\mathrm{d}s + \sum_{e\in\mathcal{E}_h} \int_e \langle q_h \boldsymbol{n} \rangle \cdot [\![\gamma \boldsymbol{v}_h]\!] \,\mathrm{d}s, \\
B_h(\boldsymbol{v}_h, q_h) &:= b(\boldsymbol{v}_h, q_h) - \sum_{e\in\mathcal{E}_h} \int_e \langle q_h \boldsymbol{n} \rangle \cdot [\![\gamma \boldsymbol{v}_h]\!] \,\mathrm{d}s,
\end{aligned}$$

for all $\boldsymbol{w}_h, \boldsymbol{v}_h \in \mathbf{V}_h$ and $q_h \in Q_h$. Here, α_d and δ are penalty parameters independent of h. In general, $\delta = (d-1)^{-1}$ is commonly used in interior penalty methods.

For the sake of our forthcoming analysis, we introduce the following discrete norms in $\mathbf{V}(h)$, which are naturally associated with the bilinear form $c_h(\cdot,\cdot)$:

$$\|\!|\boldsymbol{v}_h|\!\|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} |\boldsymbol{v}_h|_{1,T}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-\delta} \left\|[\![\boldsymbol{v}_h]\!]_e\right\|_{0,e}^2, \quad \|\!|\boldsymbol{v}_h|\!\|_{2,h}^2 := \|\!|\boldsymbol{v}_h|\!\|_{1,h}^2 + \sum_{T\in\mathcal{T}_h} h_T^2 |\boldsymbol{v}_h|_{2,T}^2,$$

and we note that $\|\!|\cdot|\!\|_{1,h}$ and $\|\!|\cdot|\!\|_{2,h}$ are equivalent on $\mathbf{V}_h$. Moreover, we also have the following discrete Poincaré-Friedrichs inequality (see [12, pp. 457])

$$\|\boldsymbol{v}_h\|_{0,\Omega} \leq C \|\!|\boldsymbol{v}_h|\!\|_{2,h} \quad \forall \boldsymbol{v}_h \in \mathbf{V}_h, \tag{5.12}$$

and as in e.g. [12], we can use Cauchy-Schwarz inequality and the definition of γ to readily obtain

$$\frac{1}{h_e^{\delta/2}} \left\|[\![\gamma \boldsymbol{v}_h]\!]_e\right\|_{0,e} \leq \left(\frac{1}{h_e^{\delta}} \int_e [\![\boldsymbol{v}_h]\!]_e^2 \,\mathrm{d}s \right)^{1/2}. \tag{5.13}$$

Lemma 5.2.2. *The bilinear forms defined above possess the following properties:*

i) The bilinear form $\hat{A}_h(\cdot,\cdot)$ is continuous and coercive, that is, there exists a constant $C > 0$, independent of h, such that

$$|\hat{A}_h(\boldsymbol{v},\boldsymbol{w})| \leq C \|\boldsymbol{v}\|_{0,\Omega} \|\boldsymbol{w}\|_{0,\Omega}, \ \forall \boldsymbol{v},\boldsymbol{w} \in \mathbf{V}(h), \tag{5.14}$$

$$\hat{A}_h(\boldsymbol{v}_h,\boldsymbol{v}_h) \geq C \|\boldsymbol{v}_h\|_{0,\Omega}^2, \ \forall \boldsymbol{v}_h \in \mathbf{V}_h. \tag{5.15}$$

In addition, we have the following estimate

$$|\hat{A}_h(\boldsymbol{v}_h,\boldsymbol{z}_h) - \hat{A}_h(\boldsymbol{z}_h,\boldsymbol{v}_h)| \leq Ch \|\!|\boldsymbol{v}_h|\!\|_{2,h} \|\!|\boldsymbol{z}_h|\!\|_{2,h} \quad \forall \boldsymbol{v}_h,\boldsymbol{z}_h \in \mathbf{V}_h. \tag{5.16}$$

ii) For the non-symmetric bilinear form $c_h(\cdot,\cdot)$ it holds that

$$|c_h(\boldsymbol{v},\boldsymbol{w})| \leq C \|\!|\boldsymbol{v}|\!\|_{2,h} \|\!|\boldsymbol{w}|\!\|_{2,h} \qquad \forall \boldsymbol{v},\boldsymbol{w} \in \mathbf{V}(h), \tag{5.17}$$

$$c_h(\boldsymbol{v}_h,\boldsymbol{v}_h) \geq C \|\!|\boldsymbol{v}_h|\!\|_{2,h}^2 \qquad \forall \boldsymbol{v}_h \in \mathbf{V}_h, \tag{5.18}$$

$$|c_h(\boldsymbol{v}_h,\boldsymbol{z}_h) - c_h(\boldsymbol{z}_h,\boldsymbol{v}_h)| \leq Ch \|\!|\boldsymbol{v}_h|\!\|_{2,h} \|\!|\boldsymbol{z}_h|\!\|_{2,h} \qquad \forall \boldsymbol{v}_h,\boldsymbol{z}_h \in \mathbf{V}_h, \tag{5.19}$$

where for (5.18), $\alpha_d > 0$ is assumed sufficiently large.

iii) The choice of approximation spaces $\mathbf{V}_h$ and Q_h for velocity and pressure, respec-

tively, yields the following inf-sup condition

$$\sup_{\boldsymbol{v}_h \in \mathbf{V}_h \setminus \{\mathbf{0}\}} \frac{B_h(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{2,h}} \geq \beta_1 \|q_h\|_{0,\Omega}, \tag{5.20}$$

where $\beta_1 > 0$ *is independent of* h*.*

iv) The bilinear form $C_h(\cdot,\cdot)$ *satisfies for all* $\boldsymbol{v}, \boldsymbol{w} \in \mathbf{V}(h)$, $q \in L_0^2(\Omega)$, *and* $q_h \in Q_h$

$$|C_h(\boldsymbol{v}, q)| \leq C \|\boldsymbol{v}\|_{2,h} \left(\|q\|_{0,\Omega} + \Big(\sum_{T \in \mathcal{T}_h} h^2 |q|_{1,T}^2 \Big)^{1/2} \right), \tag{5.21}$$

$$C_h(\boldsymbol{v}, q_h) = -B_h(\boldsymbol{v}, q_h). \tag{5.22}$$

Proof. For i) it suffices to apply the definition of $\hat{A}_h(\cdot,\cdot)$, together with relation (5.3), and the norm equivalence between $\|\cdot\|_{0,h}$ and $\|\cdot\|_{0,\Omega}$. Results in ii) have been established in [52] and [9], whereas proofs for iii)-iv) can be found in [82]. □

5.2.2 Control discretization

Let $\mathbf{U}_h \subseteq \mathbf{L}^2(\Omega)$ denote the discrete control space, and let us introduce the discrete admissible space for the control field as $\mathbf{U}_{h,\text{ad}} = \mathbf{U}_h \cap \mathbf{U}_{\text{ad}}$. Three approaches are outlined in what follows.

Variational discretization. We recall that in the so-called variational approach (cf. [42]), control variables are not discretized explicitly, that is, one simply takes $\mathbf{U}_h = \mathbf{L}^2(\Omega)$ and in this case the discrete and continuous admissible spaces $\mathbf{U}_{h,\text{ad}}$ and $\mathbf{U}_{\text{ad}}$ coincide. Induced discretization errors using this method will be postponed to Section 5.3.1.

In contrast, it is possible to fully discretize the control field. We focus on the two lowest order cases.

Piecewise linear control discretization. Here we approximate the control variable with the similar elements as those employed for the state and co-state velocity approximation. That is,

$$\mathbf{U}_h^1 = \{\boldsymbol{u}_h \in \mathbf{L}^2(\Omega) : \boldsymbol{u}_h|_T \in \mathbf{P}_1(T) \quad \forall T \in \mathcal{T}_h\}.$$

We note that the state velocity space $\mathbf{V}_h$ coincides with the control space $\mathbf{U}_h^1$ in the case of homogeneous Neumann boundary conditions, whereas for Dirichlet boundary data, we have $\mathbf{V}_h \subset \mathbf{U}_h^1$.

Piecewise constant discretization. In this case, the discrete control space is defined as

$$\mathbf{U}_h^0 = \{\boldsymbol{u}_h \in \mathbf{L}^2(\Omega) : \boldsymbol{u}_h|_T \in \mathbf{P}_0(T) \quad \forall T \in \mathcal{T}_h\}.$$

The convergence properties associated to the above two approaches (piecewise linear and piecewise constant) will be derived in Section 5.3.2, but already at this point we can apply Lemma 5.2.2 along with the Babuška-Brezzi theory for saddle point problems to ensure the unique solvability of (5.10)-(5.11), for a fixed control $\boldsymbol{u}_h$.

Using relation (5.22), the DFV approximation of the continuous optimal system (5.1)-(5.2) can be summarized as: Find $(\boldsymbol{y}_h, p_h, \boldsymbol{w}_h, r_h, \boldsymbol{u}_h) \in \mathbf{V}_h \times Q_h \times \mathbf{V}_h \times Q_h \times \mathbf{U}_{h,\text{ad}}$ such that

$$\hat{A}_h(\boldsymbol{y}_h, \boldsymbol{v}_h) + c_h(\boldsymbol{y}_h, \boldsymbol{v}_h) - B_h(\boldsymbol{v}_h, p_h) = (\boldsymbol{u}_h + \boldsymbol{f}, \gamma\boldsymbol{v}_h)_{0,\Omega}, \ \forall \boldsymbol{v}_h \in \mathbf{V}_h, \tag{5.23}$$

$$B_h(\boldsymbol{y}_h, \phi_h) = 0, \ \forall \phi_h \in Q_h, \tag{5.24}$$

$$\hat{A}_h(\boldsymbol{w}_h, \boldsymbol{z}_h) + c_h(\boldsymbol{w}_h, \boldsymbol{z}_h) + B_h(\boldsymbol{z}_h, r_h) = (\boldsymbol{y}_h - \boldsymbol{y}_d, \gamma\boldsymbol{z}_h)_{0,\Omega}, \ \forall \boldsymbol{z}_h \in \mathbf{V}_h, \tag{5.25}$$

$$B_h(\boldsymbol{w}_h, \psi_h) = 0, \ \forall \psi_h \in Q_h, \tag{5.26}$$

$$(\boldsymbol{w}_h + \lambda \boldsymbol{u}_h, \tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h)_{0,\Omega} \geq 0, \ \forall \tilde{\boldsymbol{u}}_h \in \mathbf{U}_{h,\text{ad}}. \tag{5.27}$$

5.3 Errror estimates

In this Section, we provide *a priori* error estimates for DFV approximations of the state and adjoint equations, and for the three control discretization approaches outlined in Section 5.2.2.

For a given arbitrary $\boldsymbol{u}$, let the pair $(\boldsymbol{y}_h(\boldsymbol{u}), p_h(\boldsymbol{u}))$ be the solution of the following auxiliary problem for all $\boldsymbol{v}_h \in \mathbf{V}_h$ and $\phi_h \in Q_h$:

$$\hat{A}_h(\boldsymbol{y}_h(\boldsymbol{u}), \boldsymbol{v}_h) + c_h(\boldsymbol{y}_h(\boldsymbol{u}), \boldsymbol{v}_h) - B_h(\boldsymbol{v}_h, p_h(\boldsymbol{u})) = (\boldsymbol{u} + \boldsymbol{f}, \gamma\boldsymbol{v}_h)_{0,\Omega}, \tag{5.28}$$

$$B_h(\boldsymbol{y}_h(\boldsymbol{u}), \phi_h) = 0. \tag{5.29}$$

Similarly, we assume that for an arbitrary $\boldsymbol{y}$, let $(\boldsymbol{w}_h(\boldsymbol{y}), r_h(\boldsymbol{y}))$ be the solution of

$$\hat{A}_h(\boldsymbol{w}_h(\boldsymbol{y}), \boldsymbol{z}_h) + c_h(\boldsymbol{w}_h(\boldsymbol{y}), \boldsymbol{z}_h) + B_h(\boldsymbol{z}_h, r_h(\boldsymbol{y})) = (\boldsymbol{y} - \boldsymbol{y}_d, \gamma \boldsymbol{z}_h)_{0,\Omega}, \tag{5.30}$$

$$B_h(\boldsymbol{w}_h(\boldsymbol{y}), \psi_h) = 0, \tag{5.31}$$

for all $\boldsymbol{z}_h \in \mathbf{V}_h$ and $\psi_h \in Q_h$. We then proceed to decompose the errors $\boldsymbol{y} - \boldsymbol{y}_h$, $\boldsymbol{w} - \boldsymbol{w}_h$, $p - p_h$ and $r - r_h$ in the following manner:

$$\boldsymbol{y} - \boldsymbol{y}_h = \boldsymbol{y} - \boldsymbol{y}_h(\boldsymbol{u}) + \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h, \boldsymbol{w} - \boldsymbol{w}_h = \boldsymbol{w} - \boldsymbol{w}_h(\boldsymbol{y}) + \boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h, \tag{5.32}$$

$$p - p_h = p - p_h(\boldsymbol{u}) + p_h(\boldsymbol{u}) - p_h,\ r - r_h = r - r_h(\boldsymbol{y}) + r_h(\boldsymbol{y}) - r_h. \tag{5.33}$$

Noting that $\boldsymbol{y}_h = \boldsymbol{y}_h(\boldsymbol{u}_h)$, $p_h = p_h(\boldsymbol{u}_h)$, $\boldsymbol{w}_h = \boldsymbol{w}_h(\boldsymbol{y}_h)$, and $r_h = r_h(\boldsymbol{y}_h)$, the following intermediate result is established.

Lemma 5.3.1. *Let $(\boldsymbol{y}_h(\boldsymbol{u}), p_h(\boldsymbol{u}))$ and $(\boldsymbol{w}_h(\boldsymbol{y}), r_h(\boldsymbol{y}))$ be the solutions of equations (5.28)-(5.29) and (5.30)-(5.31), respectively. Then there exists a positive constant C independent of mesh size h such that the following estimates hold*

$$\|\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h\|_{2,h} + \|p_h(\boldsymbol{u}) - p_h\|_{0,\Omega} \leq C \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}, \tag{5.34}$$

$$\|\boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h\|_{2,h} + \|r_h(\boldsymbol{y}) - r_h\|_{0,\Omega} \leq C \|\boldsymbol{y} - \boldsymbol{y}_h\|_{0,\Omega}. \tag{5.35}$$

Proof. Subtracting equations (5.23) and (5.24) from (5.28) and (5.29), respectively, we have that the following relations hold for all $\boldsymbol{v}_h \in \mathbf{V}_h$ and $\phi_h \in Q_h$

$$\begin{aligned} \hat{A}_h(\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h, \boldsymbol{v}_h) + c_h(\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h, \boldsymbol{v}_h) - B_h(\boldsymbol{v}_h, p_h(\boldsymbol{u}) - p_h) \\ = (\boldsymbol{u} - \boldsymbol{u}_h, \gamma \boldsymbol{v}_h)_{0,\Omega}, \\ B_h(\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h, \phi_h) = 0. \end{aligned} \tag{5.36}$$

Adding the above two equations after choosing $\boldsymbol{v}_h = \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h$ and $\phi_h = p_h(\boldsymbol{u}) - p_h$ implies that

$$\begin{aligned} \hat{A}_h(\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h, \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h) + c_h(\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h, \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h) \\ = (\boldsymbol{u} - \boldsymbol{u}_h, \gamma(\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h))_{0,\Omega}. \end{aligned}$$

In turn, using the coercivity of $\hat{A}_h(\cdot,\cdot)$ and $c_h(\cdot,\cdot)$ in combination with (5.9) and (5.12), we obtain

$$
\begin{aligned}
\|\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h\|_{0,\Omega}^2+\|\!|\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h|\!\|_{2,h}^2 &\leq C\big(\boldsymbol{u}-\boldsymbol{u}_h,\gamma(\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h)\big)_{0,\Omega},\\
&\leq C\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}\,\|\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h\|_{0,\Omega}\,,\\
&\leq C\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}\,\|\!|\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h|\!\|_{2,h}\,,
\end{aligned}
$$

which on dropping the first term of above relation, readily yields the bound

$$
\|\!|\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h|\!\|_{2,h}\leq C\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}\,. \tag{5.37}
$$

On the other hand, applying the inf-sup condition (5.20), using (5.36), the boundedness of $\hat{A}_h(\cdot,\cdot)$ and $c_h(\cdot,\cdot)$, along with (5.37), we realize that

$$
\begin{aligned}
&\|p_h-p_h(\boldsymbol{u})\|_{0,\Omega}\\
&\leq\frac{1}{\beta_1}\sup_{\boldsymbol{v}_h\in\mathbf{V}_h\setminus\{\mathbf{0}\}}\frac{B_h(v_h,p_h-p_h(\boldsymbol{u}))}{\|\!|\boldsymbol{v}_h|\!\|_{2,h}},\\
&=\frac{1}{\beta_1}\sup_{\boldsymbol{v}_h\in\mathbf{V}_h\setminus\{\mathbf{0}\}}\frac{\hat{A}_h(\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h,\boldsymbol{v}_h)+c_h(\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_h,\boldsymbol{v}_h)+(\boldsymbol{u}_h-\boldsymbol{u},\gamma\boldsymbol{v}_h)_{0,\Omega}}{\|\!|\boldsymbol{v}_h|\!\|_{2,h}}\\
&\leq C\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}\,.
\end{aligned} \tag{5.38}
$$

Notice that relations (5.37) and (5.38) imply, in particular, that (5.34) holds. Next, we subtract equations (5.25) and (5.26) from (5.30) and (5.31), respectively, and test the result against $\boldsymbol{z}_h=\boldsymbol{w}_h(\boldsymbol{y})-\boldsymbol{w}_h$ and $\psi_h=r_h(\boldsymbol{y})-r_h$, which yields the second desired result (5.35) after repeating the same steps as above. □

Lemma 5.3.2. *Under the assumptions $\mu\in W^{2,\infty}(\Omega)$ and $\boldsymbol{u},\boldsymbol{f},\boldsymbol{y}_d\in\mathbf{H}^1(\Omega)$, we have that*

$$
\|\!|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})|\!\|_{2,h}+\|p-p_h(u)\|_{0,\Omega}=\mathcal{O}(h), \tag{5.39}
$$

$$
\|\!|\boldsymbol{w}-\boldsymbol{w}_h(\boldsymbol{y})|\!\|_{2,h}+\|r-r_h(y)\|_{0,\Omega}=\mathcal{O}(h), \tag{5.40}
$$

$$
\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\|_{0,\Omega}+\|\boldsymbol{w}-\boldsymbol{w}_h(\boldsymbol{y})\|_{0,\Omega}=\mathcal{O}(h^2). \tag{5.41}
$$

Proof. We proceed analogously to the proof of [32, Theorem 3.1] and directly apply

Lemma 5.2.2 to readily derive the following estimates:

$$\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\|_{2,h} + \|p-p_h(u)\|_{0,\Omega} \leq Ch\left(\|\boldsymbol{y}\|_{2,\Omega} + \|p\|_{1,\Omega}\right),$$
$$\|\boldsymbol{w}-\boldsymbol{w}_h(\boldsymbol{y})\|_{2,h} + \|r-r_h(y)\|_{0,\Omega} \leq Ch\left(\|\boldsymbol{w}\|_{2,\Omega} + \|r\|_{1,\Omega}\right).$$

Next, the derivation of $\mathbf{L}^2$-estimates for $\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})$ and $\boldsymbol{w}-\boldsymbol{w}_h(\boldsymbol{y})$ follows an Aubin-Nitsche duality argument. Let us consider the dual problem: find $(\boldsymbol{z},\rho) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\mathbf{K}^{-1}(\boldsymbol{x}) - \mathbf{div}(\mu(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{z}) - \rho\mathbf{I}) = \boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}) \quad \text{in } \Omega, \tag{5.42}$$
$$\operatorname{div} \boldsymbol{z} = 0 \quad \text{in } \Omega, \tag{5.43}$$
$$\boldsymbol{z} = 0 \quad \text{on } \partial\Omega,$$

which is uniquely solvable and moreover the following $\mathbf{H}^2(\Omega)\times H^1(\Omega)$-regularity property is satisfied:

$$\|\boldsymbol{z}\|_{2,\Omega} + \|\rho\|_{1,\Omega} \leq \|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\|_{0,\Omega}. \tag{5.44}$$

Let us denote by $\boldsymbol{z}_I \in \mathbf{V}_h$ the usual continuous piecewise linear interpolant of $\boldsymbol{z}$, satisfying the following approximation properties:

$$\|\boldsymbol{z}-\boldsymbol{z}_I\|_{2,h} \leq Ch\|\boldsymbol{z}\|_{2,\Omega} \quad \text{and} \quad \|\boldsymbol{z}-\boldsymbol{z}_I\|_{0,\Omega} \leq Ch^2\|\boldsymbol{z}\|_{2,\Omega}. \tag{5.45}$$

Also, let Π_1 denote the L^2-projection from $L_0^2(\Omega)$ to Q_h, satisfying

$$\|\rho-\Pi_1\rho\|_{0,\Omega} \leq Ch\|\rho\|_{1,\Omega}.$$

Multiplying (5.42) by $\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})$, integrating by parts, and using the fact that $[\![\boldsymbol{\varepsilon}(\boldsymbol{z})\boldsymbol{n}]\!]_e = \mathbf{0}$ and $[\![\rho]\!]_e = 0$, we can obtain

$$\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\|_{0,\Omega}^2 = A_h^s(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{z}) + c_h^s(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{z}) - b_h^s(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\rho), \tag{5.46}$$

where the auxiliary bilinear forms adopt the following expressions

$$
\begin{aligned}
A_h^s(\boldsymbol{w}_h,\boldsymbol{v}_h) &:= (\mathbf{K}^{-1}(\boldsymbol{x})\boldsymbol{w}_h,\boldsymbol{v}_h)_{0,\Omega},\\
b_h^s(\boldsymbol{v}_h,q_h) &:= b(\boldsymbol{v}_h,q_h) + \sum_{e\in\mathcal{E}_h}\int_e \{q_h\boldsymbol{n}\}_e\cdot[\![\boldsymbol{v}_h]\!]_e\,\mathrm{d}s,\\
c_h^s(\boldsymbol{w}_h,\boldsymbol{v}_h) &:= c(\boldsymbol{w}_h,\boldsymbol{v}_h) - \sum_{e\in\mathcal{E}_h}\int_e \{\mu(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{w}_h)\boldsymbol{n}\}_e\cdot[\![\boldsymbol{v}_h]\!]_e\,\mathrm{d}s\\
&\qquad - \sum_{e\in\mathcal{E}_h}\int_e \{\mu(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\boldsymbol{n}\}_e\cdot[\![\boldsymbol{w}_h]\!]_e\,\mathrm{d}s + \sum_{e\in\mathcal{E}_h}\int_e \frac{\alpha_d}{h_e^\delta}[\![\boldsymbol{w}_h]\!]_e\cdot[\![\boldsymbol{v}_h]\!]_e\,\mathrm{d}s.
\end{aligned}
$$

Since $\boldsymbol{z}_I\in\mathbf{V}_h$ is a continuous interpolant of $\boldsymbol{z}$, we note that the pair $(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}), p-p_h(\boldsymbol{u}))$ will be a solution of the following problem

$$
\hat{A}_h(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{z}_I) + c_h(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{z}_I) + C_h(\boldsymbol{z}_I, p-p_h(\boldsymbol{u})) = 0, \tag{5.47}
$$

$$
B_h(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\Pi_1\rho) = 0. \tag{5.48}
$$

Using the definition of $c_h(\cdot,\cdot)$ and $C_h(\cdot,\cdot)$ we can assert that

$$
C_h(\boldsymbol{z}_I, p-p_h(\boldsymbol{u})) = -(\operatorname{div}\boldsymbol{z}_I, p-p_h(\boldsymbol{u}))_{\mathcal{T}_h} - (\nabla p, \boldsymbol{z}_I-\gamma\boldsymbol{z}_I)_{\mathcal{T}_h}, \tag{5.49}
$$

where the inner product over the primal mesh is understood as the sum of the inner products over each element in $\mathcal{T}_h$. On subtracting equation (5.47) from the sum of equations (5.46) and (5.48), and using (5.49), it follows that

$$
\begin{aligned}
&\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\|_{0,\Omega}^2\\
&= \underbrace{\Big[A_h^s(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{z}) - \hat{A}_h(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{z}_I)\Big]}_{R_1} + \underbrace{c_h^s(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{z}-\boldsymbol{z}_I)}_{R_2}\\
&+ \underbrace{\left[c_h^s(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{z}_I) - c_h(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{z}_I) + \sum_{T\in\mathcal{T}_h}\int_T(\boldsymbol{z}_I-\gamma\boldsymbol{z}_I)\cdot\nabla p\,\mathrm{d}\boldsymbol{x}\right]}_{R_3}\\
&+ \underbrace{(p-p_h(\boldsymbol{u}),\operatorname{div}\boldsymbol{z}_I)_{0,\Omega}}_{R_4} - \underbrace{b_h^s(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\rho) + B_h(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\Pi_1\rho)}_{R_5}.
\end{aligned} \tag{5.50}
$$

The estimation of R_1 relies on (5.3), (5.45), the self-adjointness and approximation properties of γ, and (5.44). This gives

$$\begin{aligned}R_1 &\le |(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\mathbf{K}^{-1}(\boldsymbol{x})\boldsymbol{z})_{0,\Omega}-(\mathbf{K}^{-1}(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})),\gamma\boldsymbol{z}_I)_{0,\Omega}|\\ &\le k_2|(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u}),\boldsymbol{z}-\boldsymbol{z}_I)_{0,\Omega}+(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})-\gamma(\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})),\boldsymbol{z}_I)_{0,\Omega}|\\ &\le C(h^2\left\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\right\|_{0,\Omega}\left\|\boldsymbol{z}\right\|_{2,\Omega}+h\left\|\!\left|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\right|\!\right\|_{2,h}\left\|\boldsymbol{z}_I\right\|_{0,\Omega})\\ &\le Ch^2(\left\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\right\|_{0,\Omega}^2+\left\|\boldsymbol{y}\right\|_{2,\Omega}\left\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\right\|_{0,\Omega}),\end{aligned}$$

where the last inequality follows from (5.39). For the second term we employ the definition of $c_h(\cdot,\cdot)$, and relations (5.45),(5.44) to verify that

$$R_2 \le Ch^2\left\|\boldsymbol{y}\right\|_{2,\Omega}\left\|\boldsymbol{z}\right\|_{2,\Omega} \le Ch^2\left\|\boldsymbol{y}\right\|_{2,\Omega}\left\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\right\|_{0,\Omega}.$$

Bounds for the remaining terms can be obtained following the proof of [52, Theorem 3.4] and [32, Theorem 3.2], as follows

$$\begin{aligned}R_3 &\le Ch^2\left[\left\|\boldsymbol{y}\right\|_{2,\Omega}+\left\|\boldsymbol{u}\right\|_{1,\Omega}+\left\|\boldsymbol{f}\right\|_{1,\Omega}\right]\left\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\right\|_{0,\Omega},\\ R_4 &\le |(p-p_h(\boldsymbol{u}),\operatorname{div}(\boldsymbol{z}-\boldsymbol{z}_I))_{0,\Omega}| \le Ch^2\left\|p\right\|_{1,\Omega}\left\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\right\|_{0,\Omega},\\ R_5 &\le Ch^2\left\|\boldsymbol{y}\right\|_{2,\Omega}\left\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\right\|_{0,\Omega}.\end{aligned}$$

Combining the five estimates above with (5.50), we straightforwardly obtain

$$\left\|\boldsymbol{y}-\boldsymbol{y}_h(\boldsymbol{u})\right\|_{0,\Omega} \le Ch^2\left[\left\|\boldsymbol{y}\right\|_{2,\Omega}+\left\|p\right\|_{1,\Omega}+\left\|\boldsymbol{u}\right\|_{1,\Omega}+\left\|\boldsymbol{f}\right\|_{1,\Omega}\right],$$

and very much in the same way, one arrives at

$$\left\|\boldsymbol{w}-\boldsymbol{w}_h(\boldsymbol{y})\right\|_{0,\Omega} \le Ch^2\left[\left\|\boldsymbol{w}\right\|_{2,\Omega}+\left\|r\right\|_{1,\Omega}+\left\|\boldsymbol{y}\right\|_{1,\Omega}+\left\|\boldsymbol{y}_d\right\|_{1,\Omega}\right].$$

□

Now, for a given control $\boldsymbol{u}$, let $(\boldsymbol{w}_h(\boldsymbol{u}),r_h(\boldsymbol{u}))$ be the solution of

$$\begin{aligned}\hat{A}_h(\boldsymbol{w}_h(\boldsymbol{u}),\boldsymbol{z}_h)+c_h(\boldsymbol{w}_h(\boldsymbol{u}),\boldsymbol{z}_h)+B_h(\boldsymbol{z}_h,r_h(\boldsymbol{u})) &= (\boldsymbol{y}_h(\boldsymbol{u})-\boldsymbol{y}_d,\gamma\boldsymbol{z}_h)_{0,\Omega} \quad \forall\boldsymbol{z}_h\in\mathbf{V}_h,\\ B_h(\boldsymbol{w}_h(\boldsymbol{u}),\psi_h) &= 0 \quad \forall\psi_h\in Q_h,\end{aligned}$$

and notice that similar arguments as those appearing in the proof of Lemma 5.3.2 and in the derivation of the estimate $\|\boldsymbol{y} - \boldsymbol{y}_h(\boldsymbol{u})\|_{0,\Omega} = \mathcal{O}(h^2)$, will readily lead to

$$\|\boldsymbol{w} - \boldsymbol{w}_h(\boldsymbol{u})\|_{0,\Omega} = \mathcal{O}(h^2). \tag{5.51}$$

The following result plays a vital role in deriving error estimates of the control, state and co-state variables. Its proof is similar to that in [59, Theorem 4.1].

Lemma 5.3.3. *Under the assumptions $\mu \in W^{2,\infty}(\Omega)$ and $\boldsymbol{u}, \boldsymbol{f}, \boldsymbol{y}_d \in \mathbf{H}^1(\Omega)$, we have the following estimate*

$$(\boldsymbol{w} - \boldsymbol{w}_h, \boldsymbol{u}_h - \boldsymbol{u})_{0,\Omega} \leq Ch^2 \|\boldsymbol{u}_h - \boldsymbol{u}\|_{0,\Omega} + Ch \|\boldsymbol{u}_h - \boldsymbol{u}\|_{0,\Omega}^2 . \tag{5.52}$$

Proof. We split $(\boldsymbol{w} - \boldsymbol{w}_h, \boldsymbol{u}_h - \boldsymbol{u})_{0,\Omega}$ as

$$\begin{aligned}(\boldsymbol{w} - \boldsymbol{w}_h, \boldsymbol{u}_h - \boldsymbol{u})_{0,\Omega} =&(\boldsymbol{w} - \boldsymbol{w}_h(y), \boldsymbol{u}_h - \boldsymbol{u})_{0,\Omega} + (\boldsymbol{w}_h(y) - \boldsymbol{w}_h - \gamma(\boldsymbol{w}_h(y) \\ &- \boldsymbol{w}_h), \boldsymbol{u}_h - \boldsymbol{u})_{0,\Omega} + (\gamma(\boldsymbol{w}_h(y) - \boldsymbol{w}_h), \boldsymbol{u}_h - \boldsymbol{u})_{0,\Omega}.\end{aligned} \tag{5.53}$$

Then, using the approximation property of γ together with Lemmas 5.3.1 and 5.3.2 implies

$$\begin{aligned}&(\boldsymbol{w} - \boldsymbol{w}_h(y), \boldsymbol{u}_h - \boldsymbol{u})_{0,\Omega} + (\boldsymbol{w}_h(y) - \boldsymbol{w}_h - \gamma(\boldsymbol{w}_h(y) - \boldsymbol{w}_h), \boldsymbol{u}_h - \boldsymbol{u})_{0,\Omega} \\ &\quad\leq \|\boldsymbol{w} - \boldsymbol{w}_h(y)\|_{0,\Omega} \|\boldsymbol{u}_h - \boldsymbol{u}\|_{0,\Omega} + Ch \|\boldsymbol{y} - \boldsymbol{y}_h\|_{0,\Omega} \|\boldsymbol{u}_h - \boldsymbol{u}\|_{0,\Omega} \\ &\quad\leq Ch^2 \|\boldsymbol{u}_h - \boldsymbol{u}\|_{0,\Omega} + Ch(\|\boldsymbol{y} - \boldsymbol{y}_h(\boldsymbol{u})\|_{0,\Omega} + \|\boldsymbol{u}_h - \boldsymbol{u}\|_{0,\Omega}) \|\boldsymbol{u}_h - \boldsymbol{u}\|_{0,\Omega} \\ &\quad\leq Ch^2 \|\boldsymbol{u}_h - \boldsymbol{u}\|_{0,\Omega} + Ch \|\boldsymbol{u}_h - \boldsymbol{u}\|_{0,\Omega}^2 .\end{aligned} \tag{5.54}$$

Now we subtract (5.28) and (5.29) from (5.23) and (5.24), respectively and test the result against $\boldsymbol{v}_h = \boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h$ and $\phi_h = r_h(\boldsymbol{y}) - r_h$ to obtain the relation

$$\begin{aligned}&(\gamma(\boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h), \boldsymbol{u}_h - \boldsymbol{u})_{0,\Omega} \\ &=A_h(\boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u}), \boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h) + c_h(\boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u}), \boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h) \\ &\quad - B_h(\boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h, p_h - p_h(\boldsymbol{u})) + B(\boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u}), r_h(\boldsymbol{y}) - r_h).\end{aligned} \tag{5.55}$$

Similarly, on subtracting equations (5.25) and (5.26) from (5.30) and (5.31), respectively, and taking $\boldsymbol{z}_h = \boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u})$ and $\psi_h = p_h - p_h(\boldsymbol{u})$, we can assert that

$$\begin{aligned}
&\hat{A}_h(\boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h, \boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u})) + c_h(\boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h, \boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u})) \\
&= (\boldsymbol{y} - \boldsymbol{y}_h, \gamma(\boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u})))_{0,\Omega} - B_h(\boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u}), r_h(\boldsymbol{y}) - r_h) \\
&+ B(\boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h, p_h - p_h(\boldsymbol{u})).
\end{aligned} \tag{5.56}$$

Adding equations (5.55) and (5.56) and using the fact that $(\boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u}), \gamma(\boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u})))_{0,\Omega} \geq 0$, we arrive at

$$\begin{aligned}
&(\gamma(\boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h), \boldsymbol{u}_h - \boldsymbol{u})_{0,\Omega} \\
\leq &[\hat{A}_h(\boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u}), \boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h) - \hat{A}_h(\boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h, \boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u}))] \\
&+ [c_h(\boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u}), \boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h) - c_h(\boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h, \boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u}))] \\
&+ (\boldsymbol{y} - \boldsymbol{y}_h(\boldsymbol{u}), \gamma(\boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u})))_{0,\Omega} \\
\leq &Ch \, |||\boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u})|||_{2,h} \, |||\boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h|||_{2,h} + \|\boldsymbol{y} - \boldsymbol{y}_h(\boldsymbol{u})\|_{0,\Omega} \, |||\boldsymbol{y}_h - \boldsymbol{y}_h(\boldsymbol{u})|||_{2,h} \,,
\end{aligned}$$

where we have used relations (5.9), (5.12), (5.16) and (5.19). An application of Lemmas 5.3.1 and 5.3.2 in the above inequality implies

$$(\gamma(\boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h), \boldsymbol{u}_h - \boldsymbol{u})_{0,\Omega} \leq Ch \, \|\boldsymbol{u}_h - \boldsymbol{u}\|_{0,\Omega}^2 + Ch^2 \, \|\boldsymbol{u}_h - \boldsymbol{u}\|_{0,\Omega} \,. \tag{5.57}$$

Inserting the estimates of (5.54) and (5.57) in (5.53), we get the required result. □

5.3.1 Error estimates under variational discretization

Theorem 5.3.4. *Let $(\boldsymbol{y}_h, \boldsymbol{w}_h)$ be DFV approximations of $(\boldsymbol{y}, \boldsymbol{w})$ and let $\boldsymbol{u}_h$ denote a variational discretization of $\boldsymbol{u}$. Then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} = \mathcal{O}(h^2), \tag{5.58}$$

$$\|\boldsymbol{y} - \boldsymbol{y}_h\|_{0,\Omega} = \mathcal{O}(h^2), \tag{5.59}$$

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{0,\Omega} = \mathcal{O}(h^2). \tag{5.60}$$

Proof. We recall the continuous variational inequality

$$(\boldsymbol{w} + \lambda \boldsymbol{u}, \tilde{\boldsymbol{u}} - \boldsymbol{u})_{0,\Omega} \geq 0 \quad \forall \tilde{\boldsymbol{u}} \in \mathbf{U}_{\mathrm{ad}}, \tag{5.61}$$

and the discrete variational inequality under variational discretization

$$(\boldsymbol{w}_h + \lambda \boldsymbol{u}_h, \tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h)_{0,\Omega} \geq 0 \quad \forall \tilde{\boldsymbol{u}}_h \in \mathbf{U}_{\text{ad}}. \tag{5.62}$$

Choosing $\tilde{\boldsymbol{u}} = \boldsymbol{u}_h$ and $\tilde{\boldsymbol{u}}_h = \boldsymbol{u}$ in (5.61) and (5.62), respectively, and adding, yields

$$(\boldsymbol{w} + \lambda \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u})_{0,\Omega} + (\boldsymbol{w}_h + \lambda \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h)_{0,\Omega} \geq 0,$$

and rearranging terms, we get

$$\lambda \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}^2 \leq (\boldsymbol{w} - \boldsymbol{w}_h, \boldsymbol{u}_h - \boldsymbol{u})_{0,\Omega}. \tag{5.63}$$

An application of (5.52) in (5.63) yields the required result (5.58). Using (5.32) and the triangle inequality together with Lemmas 5.3.1 and 5.3.2, and result (5.58), the remaining estimates (5.59)-(5.60) follow in a straightforward manner. □

5.3.2 L^2-error estimates for fully discretized controls

A discrete admissible control $\tilde{\boldsymbol{u}}_h = (\tilde{u}_{h,j})_{j=1}^d \in \mathbf{U}_{h,\text{ad}}$ is defined componentwise, and on an arbitrary $T \in \mathcal{T}_h$, as

$$\tilde{u}_{h,j} = \begin{cases} u_{a_j} & \text{if } \min_{x \in T} u_j(\boldsymbol{x}) = u_{a_j}, \\ u_{b_j} & \text{if } \max_{x \in T} u_j(\boldsymbol{x}) = u_{b_j}, \\ \tilde{I}_h u_j & \text{otherwise}, \end{cases} \tag{5.64}$$

where $\tilde{I}_h u_j$ denotes the Lagrange interpolate of u_j. To avoid ambiguity, we choose h sufficiently small so that $\min_{x \in T} u_j(\boldsymbol{x}) = u_{a_j}$ and $\max_{x \in T} u_j(\boldsymbol{x}) = u_{b_j}$ do not occur simultaneously within the same element T. Next, we proceed to group the elements in the primal mesh into three categories: $\mathcal{T}_h = \mathcal{T}_{h,1}^j \cup \mathcal{T}_{h,2}^j \cup \mathcal{T}_{h,3}^j$ with $\mathcal{T}_{h,m}^j \cap \mathcal{T}_{h,n}^j = \emptyset$ for $m \neq n$ according to the local value of $u_j(\boldsymbol{x})$ on T. These sets are defined as

$$\begin{aligned} \mathcal{T}_{h,1}^j &= \{T \in \mathcal{T}_h : u_j(\boldsymbol{x}) = u_{a_j} \ \text{ or } \ u_j(\boldsymbol{x}) = u_{b_j} \quad \forall x \in T\}, \\ \mathcal{T}_{h,2}^j &= \{T \in \mathcal{T}_h : u_{a_j} < u_j(\boldsymbol{x}) < u_{b_j} \quad \forall x \in T\}, \\ \mathcal{T}_{h,3}^j &= \mathcal{T}_h \setminus (\mathcal{T}_{h,1}^j \cup \mathcal{T}_{h,2}^j). \end{aligned}$$

Definition (5.64) implies that for any $\tilde{\boldsymbol{u}}_h \in \mathbf{U}_{h,\mathrm{ad}}$, the following relation holds (cf. [17, Lemma 2.1]):

$$(\boldsymbol{w} + \lambda \boldsymbol{u}, \tilde{\boldsymbol{u}} - \tilde{\boldsymbol{u}}_h)_{0,\Omega} \geq 0 \qquad \forall \tilde{\boldsymbol{u}} \in \mathbf{U}_{\mathrm{ad}}. \tag{5.65}$$

On the other hand, we have the following technical result, to be instrumental in the subsequent analysis: there exists a positive constant C independent of h such that

$$\sum_{j=1}^{d} \sum_{T \in \mathcal{T}_{h,3}^{j}} |T| \leq Ch. \tag{5.66}$$

We will first focus on error bounds for the control field under piecewise linear discretization. Before proceeding we state an auxiliary result, whose proof can be found in [62].

Lemma 5.3.5. *Assume that* (5.66) *holds and that* $\boldsymbol{w} \in \mathbf{W}^{1,\infty}(\Omega)$. *Then, there exists* $C > 0$ *independent of* h *such that*

$$\left|(\boldsymbol{w} + \lambda \boldsymbol{u}, \tilde{\boldsymbol{u}}_h - \boldsymbol{u})_{0,\Omega}\right| \leq \frac{C}{\lambda} h^3 \left\|\nabla \boldsymbol{w}\right\|_{\infty,\Omega}^2 ,$$

for any $\tilde{\boldsymbol{u}}_h \in \mathbf{U}_{h,\mathrm{ad}}$.

The main result in this Section is stated as follows.

Theorem 5.3.6. *Let* $\boldsymbol{u} \in \mathrm{U}_{\mathrm{ad}}$ *be the solution of* (5.1)-(5.2) *and* $\boldsymbol{u}_h \in \mathbf{U}_{h,\mathrm{ad}}$ *be the solution of* (5.23)-(5.27)*, under piecewise linear control discretization. Then*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} = \mathcal{O}(h^{3/2}).$$

Proof. Testing the continuous and discrete variational inequalities against $\boldsymbol{u}_h \in \mathbf{U}_{h,\mathrm{ad}} \subset \mathrm{U}_{\mathrm{ad}}$ and $\tilde{\boldsymbol{u}}_h \in \mathrm{U}_{h,\mathrm{ad}}$, respectively, and adding them, leads to

$$(\boldsymbol{w} + \lambda \boldsymbol{u}, \boldsymbol{u}_h - \boldsymbol{u})_{0,\Omega} + (\boldsymbol{w}_h + \lambda \boldsymbol{u}_h, \tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h)_{0,\Omega} \geq 0.$$

Addition and subtraction of $\tilde{\boldsymbol{u}}_h$ in the first term above yields

$$\lambda(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u}_h - \tilde{\boldsymbol{u}}_h)_{0,\Omega} + (\boldsymbol{w} - \boldsymbol{w}_h, \boldsymbol{u}_h - \tilde{\boldsymbol{u}}_h)_{0,\Omega} + (\boldsymbol{w} + \lambda \boldsymbol{u}, \tilde{\boldsymbol{u}}_h - \boldsymbol{u})_{0,\Omega} \geq 0,$$

and after rearranging terms we obtain

$$\begin{aligned}\lambda \|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}^2 \quad \leq \quad & \lambda(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{u}-\tilde{\boldsymbol{u}}_h)_{0,\Omega} + (\boldsymbol{w}-\boldsymbol{w}_h, \boldsymbol{u}_h - \boldsymbol{u})_{0,\Omega} \\ & + (\boldsymbol{w}-\boldsymbol{w}_h, \boldsymbol{u}-\tilde{\boldsymbol{u}}_h)_{0,\Omega} + (\boldsymbol{w}+\lambda\boldsymbol{u}, \tilde{\boldsymbol{u}}_h - \boldsymbol{u})_{0,\Omega}. \end{aligned} \quad (5.67)$$

Now, in order to estimate $\|\boldsymbol{u}-\tilde{\boldsymbol{u}}_h\|_{0,\Omega}$, we rewrite it as

$$\begin{aligned}\|\boldsymbol{u}-\tilde{\boldsymbol{u}}_h\|_{0,\Omega}^2 &= \sum_{j=1}^{d}\sum_{T\in\mathcal{T}_h} \|u_j - \tilde{u}_{j,h}\|_{0,T}^2 \\ &= \sum_{j=1}^{d}\sum_{T\in\mathcal{T}_{h,2}^j} \|u_j - \tilde{u}_{j,h}\|_{0,T}^2 + \sum_{j=1}^{d}\sum_{T\in\mathcal{T}_{h,3}^j} \|u_j - \tilde{u}_{j,h}\|_{0,T}^2 \\ &=: T_1 + T_2, \end{aligned} \quad (5.68)$$

where we have used the fact that $\tilde{u}_{j,h} = u_j$ on $\mathcal{T}_{h,1}^j$, and hence $\sum\limits_{T\in\mathcal{T}_{h,1}^j} \|u_j - \tilde{u}_{j,h}\|_{0,T}^2 = 0$, for $j = 1, \ldots, d$. In order to bound T_1 we use the relation $u_j = \frac{-1}{\lambda} w_j$ on all triangles $T \in \mathcal{T}_{h,2}^j$, to obtain

$$\sum_{j=1}^{d}\sum_{T\in\mathcal{T}_{h,2}^j} \left\|u_i - \tilde{I}_h u_i\right\|_{0,T}^2 \leq Ch^4 \sum_{j=1}^{d}\sum_{T\in\mathcal{T}_{h,2}^j} \|\nabla^2 u_j\|_{0,T}^2 \leq \frac{C}{\lambda^2} h^4 \sum_{j=1}^{d} \left\|\nabla^2 w_i\right\|^2,$$

whereas for T_2, we employ the projection property (5.6) together with Assumption 5.66 to get

$$\begin{aligned}\sum_{j=1}^{d}\sum_{T\in\mathcal{T}_{h,3}^j} \left\|u_j - \tilde{I}_h u_j\right\|_{0,T}^2 &\leq C \sum_{j=1}^{d}\sum_{T\in\mathcal{T}_{h,3}^j} |T| \left\|u_j - \tilde{I}_h u_j\right\|_{L^\infty(T)}^2 \\ &\leq Ch^3 \sum_{j=1}^{d} \|\nabla u_j\|_{L^\infty(\Omega)}^2 \leq \frac{C}{\lambda^2} h^3 \sum_{j=1}^{d} \|\nabla w_j\|_{L^\infty(\Omega)}^2. \end{aligned}$$

Inserting the bounds of T_1 and T_2 in (5.68) we arrive at

$$\|\boldsymbol{u}-\tilde{\boldsymbol{u}}_h\|_{0,\Omega} = \mathcal{O}(h^{3/2}). \quad (5.69)$$

Finally, applying Cauchy-Schwarz and Young's inequalities, the estimates (5.52), (5.69) and Lemmas 5.3.2 and 3.3.8 in (2.36), we readily obtain the required result

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega} = \mathcal{O}(h^{3/2}).$$

□

We now turn to the $\mathbf{L}^2-$error analysis for the control field under element-wise constant discretization. The main idea follows from [17], where the $\mathbf{L}^2-$projection operator $\Pi_0 : \mathbf{L}^2(\Omega) \longrightarrow \mathbf{U}_{h,0}$ is introduced, having the following property: there exists a positive constant C independent of h such that

$$\|\boldsymbol{u}-\Pi_0\boldsymbol{u}\|_{0,\Omega} \leq Ch\,\|\boldsymbol{u}\|_{1,\Omega}\,. \tag{5.70}$$

The error estimate reads as follows.

Theorem 5.3.7. *Let $\boldsymbol{u}$ be the unique solution of problem (5.1)-(5.2) and $\boldsymbol{u}_h$ be the unique control, solution of discrete problem (5.23)-(5.27) under element-wise constant discretization. Then we can obtain the following result.*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega} = \mathcal{O}(h).$$

Proof. Since $\Pi_0\mathbf{U}_{\text{ad}} \subset \mathbf{U}_{h,\text{ad}}$, the continuous and discrete optimality conditions readily imply

$$(\boldsymbol{w}+\lambda\boldsymbol{u},\boldsymbol{u}_h-\boldsymbol{u})_{0,\Omega} + (\boldsymbol{w}_h+\lambda\boldsymbol{u}_h,\Pi_0\boldsymbol{u}-\boldsymbol{u}_h)_{0,\Omega} \geq 0.$$

Adding and subtracting $\boldsymbol{u}$ and rearranging terms we obtain

$$\lambda\,\|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{0,\Omega} \leq (\boldsymbol{w}-\boldsymbol{w}_h,\boldsymbol{u}_h-\boldsymbol{u})_{0,\Omega} + (\boldsymbol{w}_h+\lambda\boldsymbol{u}_h,\Pi_0\boldsymbol{u}-\boldsymbol{u})_{0,\Omega},$$

and since Π_0 is an orthogonal projection and $\boldsymbol{u}_h \in \mathbf{U}_{h,\text{ad}}$, then the term $\lambda(\boldsymbol{u}_h,\Pi_0\boldsymbol{u}-\boldsymbol{u})_{0,\Omega}$ vanishes to give

$$\lambda\,\|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{0,\Omega} \leq (\boldsymbol{w}-\boldsymbol{w}_h,\boldsymbol{u}_h-\boldsymbol{u})_{0,\Omega} + (\boldsymbol{w}_h,\Pi_0\boldsymbol{u}-\boldsymbol{u})_{0,\Omega} =: I_1+I_2. \tag{5.71}$$

For the first term, we use (5.52)

$$I_1 \leq Ch^2 \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} + Ch \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}^2 ,$$

whereas a bound for I_2 follows from the orthogonality of Π_0:

$$\begin{aligned} I_2 = (\boldsymbol{w}_h - \Pi_0 \boldsymbol{w}_h, \Pi_0 \boldsymbol{u} - \boldsymbol{u})_{0,\Omega} &\leq \|\boldsymbol{w}_h - \Pi_0 \boldsymbol{w}_h\|_{0,\Omega} \|\Pi_0 \boldsymbol{u} - \boldsymbol{u}\|_{0,\Omega} \\ &\leq Ch \|\!|\boldsymbol{w}_h|\!\|_{2,h} \|\Pi_0 \boldsymbol{u} - \boldsymbol{u}\|_{0,\Omega} . \end{aligned}$$

It is left to show that $\boldsymbol{w}_h$ is uniformly bounded, which can be readily derived using the coercivity of the forms $\hat{A}_h(\cdot,\cdot)$ and $c_h(\cdot,\cdot)$ and the uniform boundedness of $\mathbf{U}_{h,\text{ad}}$:

$$\|\!|\boldsymbol{w}_h|\!\|_{2,h} \leq C \left(\|\boldsymbol{u}_h\|_{0,\Omega} + \|\boldsymbol{f}\|_{0,\Omega} + \|\boldsymbol{y}_d\|_{0,\Omega} \right) \leq C.$$

Therefore, substituting the bounds for I_1 and I_2 in (5.71), and using (2.30) the desired result follows. □

5.3.3 L^2-error estimates for velocity under full discretization of control

The main result in this Section is given as follows (see similar ideas, based on duality arguments also applied in [61, 67]).

Theorem 5.3.8. *Let $(\boldsymbol{y}, \boldsymbol{w})$ be the state and co-state velocities, solutions of (5.1)-(5.2), and let $(\boldsymbol{y}_h, \boldsymbol{w}_h)$ be their DFV approximations under piecewise linear (or piecewise constant) discretization of the control field. Then*

$$\|\boldsymbol{y} - \boldsymbol{y}_h\|_{0,\Omega} = \mathcal{O}(h^2), \quad \textit{and} \quad \|\boldsymbol{w} - \boldsymbol{w}_h\|_{0,\Omega} = \mathcal{O}(h^2).$$

Proof. We start by splitting the total error and applying triangle inequality as:

$$\|\boldsymbol{y} - \boldsymbol{y}_h\|_{0,\Omega} \leq \|\boldsymbol{y} - \boldsymbol{y}_h(\boldsymbol{u})\|_{0,\Omega} + \|\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})\|_{0,\Omega} + \|\boldsymbol{y}_h(\Pi_h \boldsymbol{u}) - \boldsymbol{y}_h\|_{0,\Omega} , \tag{5.72}$$

where Π_h represents the $\mathbf{L}^2-$projection operator onto the discrete control space $\mathbf{U}_h$. Next, let $(\tilde{\boldsymbol{w}}_h, \tilde{r}_h) \in \mathbf{V}_h \times Q_h$ be the unique solution of the auxiliary discrete dual

Brinkman problem for all $\tilde{\boldsymbol{z}}_h \in \mathbf{V}_h$ and $\tilde{\psi}_h \in Q_h$

$$\hat{A}_h(\tilde{\boldsymbol{w}}_h, \tilde{\boldsymbol{z}}_h) + c_h(\tilde{\boldsymbol{w}}_h, \tilde{\boldsymbol{z}}_h) - B_h(\tilde{\boldsymbol{z}}_h, \tilde{r}_h) = (\gamma \tilde{\boldsymbol{z}}_h, \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}))_{0,\Omega}, \tag{5.73}$$

$$B_h(\tilde{\boldsymbol{w}}_h, \tilde{\psi}_h) = 0. \tag{5.74}$$

We then choose $\tilde{\boldsymbol{z}}_h = \tilde{\boldsymbol{w}}_h$ and $\tilde{\psi}_h = \tilde{r}_h$ in (5.73) and (5.74), respectively, next we add the result, and we use the coercivity properties (5.15) and (5.18), to derive that

$$\|\tilde{\boldsymbol{w}}_h\|_{2,h} \leq C \|\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})\|_{0,\Omega}. \tag{5.75}$$

After testing (5.73)-(5.74) against $\tilde{\boldsymbol{z}}_h = \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})$ and $\tilde{\psi}_h = p_h(\boldsymbol{u}) - p_h(\Pi_h \boldsymbol{u})$, respectively, and adding the result, we obtain

$$\begin{aligned} &\hat{A}_h(\tilde{\boldsymbol{w}}_h, \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})) + c_h(\tilde{\boldsymbol{w}}_h, \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})) - B_h(\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}), \tilde{r}_h) \\ &- B_h(\tilde{\boldsymbol{w}}_h, p_h(\boldsymbol{u}) - p_h(\Pi_h \boldsymbol{u})) = (\gamma \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}), \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}))_{0,\Omega}. \end{aligned} \tag{5.76}$$

In addition, employing the discrete state equation for $\boldsymbol{y}_h(\boldsymbol{u})$ and $\boldsymbol{y}_h(\Pi_h \boldsymbol{u})$, we obtain

$$\begin{aligned} \hat{A}_h(\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}), \tilde{\boldsymbol{w}}_h) + c_h(\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}), \tilde{\boldsymbol{w}}_h) - B_h(\tilde{\boldsymbol{w}}_h, p_h(\boldsymbol{u}) - p_h(\Pi_h \boldsymbol{u})) \\ - B_h(\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}), \tilde{r}_h) = (\boldsymbol{u} - \Pi_h \boldsymbol{u}, \gamma \tilde{\boldsymbol{w}}_h)_{0,\Omega}. \end{aligned} \tag{5.77}$$

We then proceed to subtract (5.77) from (5.76) and to rearrange terms, to arrive at

$$\begin{aligned} &(\gamma \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}), \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}))_{0,\Omega} \\ &= \hat{A}_h(\tilde{\boldsymbol{w}}_h, \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})) - \hat{A}_h(\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}), \tilde{\boldsymbol{w}}_h) \\ &\quad + c_h(\tilde{\boldsymbol{w}}_h, \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})) - c_h(\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}), \tilde{\boldsymbol{w}}_h) \\ &\quad + (\boldsymbol{u} - \Pi_h \boldsymbol{u}, \gamma \tilde{\boldsymbol{w}}_h)_{0,\Omega}. \end{aligned}$$

Using the definition of the norm $\|\cdot\|_{0,h}$ and its equivalence with the norm $\|\cdot\|_{0,\Omega}$ we find that

$$
\begin{aligned}
&\|\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})\|_{0,\Omega}^2 \\
&\leq (\boldsymbol{u} - \Pi_h \boldsymbol{u}, \gamma \tilde{\boldsymbol{w}}_h)_{0,\Omega} + |\hat{A}_h(\tilde{\boldsymbol{w}}_h, \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})) - \hat{A}_h(\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}), \tilde{\boldsymbol{w}}_h)| \\
&\qquad + |c_h(\tilde{\boldsymbol{w}}_h, \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})) - c_h(\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}), \tilde{\boldsymbol{w}}_h)|.
\end{aligned}
$$

By virtue of properties of Π_h applied in the above inequality, we can assert that

$$
\begin{aligned}
&\|\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})\|_{0,\Omega}^2 \\
&\leq (\boldsymbol{u} - \Pi_h \boldsymbol{u}, \gamma \tilde{\boldsymbol{w}}_h - \tilde{\boldsymbol{w}}_h)_{0,\Omega} + (\boldsymbol{u} - \Pi_h \boldsymbol{u}, \tilde{\boldsymbol{w}}_h - \Pi_h \tilde{\boldsymbol{w}}_h)_{0,\Omega} \\
&\qquad + |\hat{A}_h(\tilde{\boldsymbol{w}}_h, \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})) - \hat{A}_h(\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}), \tilde{\boldsymbol{w}}_h)| \\
&\qquad + |c_h(\tilde{\boldsymbol{w}}_h, \boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})) - c_h(\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}), \tilde{\boldsymbol{w}}_h)| \\
&=: S_1 + S_2 + S_3 + S_4. \qquad (5.78)
\end{aligned}
$$

Approximation properties of γ and the $\mathbf{L}^2-$projection readily yield appropriate bounds for S_1 and S_2, respectively:

$$
\begin{aligned}
S_1 &\leq Ch \|\boldsymbol{u} - \Pi_h \boldsymbol{u}\|_{0,\Omega} \|\tilde{\boldsymbol{w}}_h\|_{2,h}, \\
S_2 &\leq Ch \|\boldsymbol{u} - \Pi_h \boldsymbol{u}\|_{0,\Omega} \|\tilde{\boldsymbol{w}}_h\|_{2,h}.
\end{aligned}
$$

Then, a direct application of (5.75) yields

$$
S_1 + S_2 \leq Ch \|\boldsymbol{u} - \Pi_h \boldsymbol{u}\|_{0,\Omega} \|\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})\|_{0,\Omega}.
$$

We next use relations (5.16), (5.19) and (5.75) to obtain

$$
\begin{aligned}
S_3 + S_4 &\leq Ch \|\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})\|_{2,h} \|\tilde{\boldsymbol{w}}_h\|_{2,h} \\
&\leq Ch \|\boldsymbol{u} - \Pi_h \boldsymbol{u}\|_{0,\Omega} \|\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})\|_{0,\Omega}.
\end{aligned}
$$

Finally, substituting the estimates for S_1, S_2, S_3 and S_4 in (5.78), one straightforwardly arrives at

$$
\|\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})\|_{0,\Omega} \leq Ch \|\boldsymbol{u} - \Pi_h \boldsymbol{u}\|_{0,\Omega}. \qquad (5.79)
$$

For the third term in (2.43) we exploit (5.12) and proceed similarly as in the proof of

Lemma 5.3.1 to obtain

$$\|\boldsymbol{y}_h(\Pi_h \boldsymbol{u}) - \boldsymbol{y}_h\|_{0,\Omega} \leq \|\!|\boldsymbol{y}_h(\Pi_h \boldsymbol{u}) - \boldsymbol{y}_h|\!\|_{2,h} \leq C \|\Pi_h \boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} . \tag{5.80}$$

Using the discrete variational inequality along with the projection property of Π_h and (5.65), we have the following relation

$$\begin{aligned}
&\lambda \|\Pi_h \boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}^2 \\
=&\lambda(\boldsymbol{u} - \boldsymbol{u}_h, \Pi_h \boldsymbol{u} - \boldsymbol{u}_h)_{0,\Omega} \\
\leq&(\boldsymbol{w} - \boldsymbol{w}_h, \boldsymbol{u}_h - \Pi_h \boldsymbol{u})_{0,\Omega} \\
=&(\boldsymbol{w} - \boldsymbol{w}_h(u), \boldsymbol{u}_h - \Pi_h \boldsymbol{u})_{0,\Omega} + (\boldsymbol{w}_h(u) - \boldsymbol{w}_h(y_h(\Pi_h u)), \boldsymbol{u}_h - \Pi_h \boldsymbol{u})_{0,\Omega} \\
&\quad + (\boldsymbol{w}_h(y_h(\Pi_h u)) - \boldsymbol{w}_h, \boldsymbol{u}_h - \Pi_h \boldsymbol{u})_{0,\Omega} \\
=&(\boldsymbol{w} - \boldsymbol{w}_h(u), \boldsymbol{u}_h - \Pi_h \boldsymbol{u})_{0,\Omega} + (\boldsymbol{w}_h(u) - \boldsymbol{w}_h(y_h(\Pi_h u)), \boldsymbol{u}_h - \Pi_h \boldsymbol{u})_{0,\Omega} \\
&\quad + (\boldsymbol{w}_h(y_h(\Pi_h u)) - \boldsymbol{w}_h - \gamma(\boldsymbol{w}_h(y_h(\Pi_h u)) - \boldsymbol{w}_h), \boldsymbol{u}_h - \Pi_h \boldsymbol{u})_{0,\Omega} \\
&\quad + (\gamma(\boldsymbol{w}_h(y_h(\Pi_h u)) - \boldsymbol{w}_h), \boldsymbol{u}_h - \Pi_h \boldsymbol{u})_{0,\Omega} \\
=:&J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{5.81}$$

Next, we use Cauchy-Schwarz inequality and (5.51) to bound the first term:

$$J_1 \leq \|\boldsymbol{w} - \boldsymbol{w}_h(u)\|_{0,\Omega} \|\boldsymbol{u}_h - \Pi_h \boldsymbol{u}\|_{0,\Omega} \leq Ch^2 \|\boldsymbol{u}_h - \Pi_h \boldsymbol{u}\|_{0,\Omega} .$$

For J_2, an application of Lemma 5.3.1 and (5.79) suffices to get

$$J_2 \leq \|\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})\|_{0,\Omega} \|\boldsymbol{u}_h - \Pi_h \boldsymbol{u}\|_{0,\Omega} \leq Ch \|\boldsymbol{u} - \Pi_h \boldsymbol{u}\|_{0,\Omega} \|\boldsymbol{u}_h - \Pi_h \boldsymbol{u}\|_{0,\Omega} .$$

To bound the third term we use the approximation propery of γ and Lemma 5.3.1

$$\begin{aligned}
J_3 &\leq Ch \|\!|\boldsymbol{w}_h(\boldsymbol{y}_h(\Pi_h \boldsymbol{u})) - \boldsymbol{w}_h|\!\|_{2,h} \|\boldsymbol{u}_h - \Pi_h \boldsymbol{u}\|_{0,\Omega} \\
&\leq Ch \|\boldsymbol{y}_h(\Pi_h \boldsymbol{u}) - \boldsymbol{y}_h\|_{0,\Omega} \|\boldsymbol{u}_h - \Pi_h \boldsymbol{u}\|_{0,\Omega} \leq Ch \|\boldsymbol{u}_h - \Pi_h \boldsymbol{u}\|_{0,\Omega}^2 .
\end{aligned}$$

Proceeding analogously to the proof of Lemma 5.3.3, using (5.16) and (5.19), the last term of the expression (5.81) can be estimated as

$$\begin{aligned} J_4 &\le \hat{A}_h(\boldsymbol{y}_h - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}), \boldsymbol{w}_h(\Pi_h \boldsymbol{u}) - w_h) - \hat{A}_h(\boldsymbol{w}_h(\Pi_h \boldsymbol{u}) - w_h, \boldsymbol{y}_h - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})) \\ &\quad + c_h(\boldsymbol{y}_h - \boldsymbol{y}_h(\Pi_h \boldsymbol{u}), \boldsymbol{w}_h(\Pi_h \boldsymbol{u}) - w_h) - c_h(\boldsymbol{w}_h(\Pi_h \boldsymbol{u}) - w_h, \boldsymbol{y}_h - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})) \\ &\le Ch \, |||\boldsymbol{y}_h - \boldsymbol{y}_h(\Pi_h \boldsymbol{u})|||_{2,h} \, |||\boldsymbol{w}_h(\Pi_h \boldsymbol{u}) - w_h|||_{2,h} \le Ch \, \|\boldsymbol{u}_h - \Pi_h \boldsymbol{u}\|_{0,\Omega}^2 . \end{aligned}$$

Plugging the bounds for J_1, J_2, J_3 and J_4 in (5.81), putting (2.49) and (5.80) into (2.43), and using interpolation estimates, along with Lemma 5.3.2; we obtain an optimal estimate for the state velocity error

$$\|\boldsymbol{y} - \boldsymbol{y}_h\|_{0,\Omega} = \mathcal{O}(h^2). \tag{5.82}$$

Finally, splitting the co-state velocity error as $\boldsymbol{w} - \boldsymbol{w}_h = \boldsymbol{w} - \boldsymbol{w}_h(\boldsymbol{y}) + \boldsymbol{w}_h(y) - \boldsymbol{w}_h$, using triangle inequality and Lemmas 5.3.1,5.3.2, and relation (5.82), we get the second desired estimate

$$\begin{aligned} \|\boldsymbol{w} - \boldsymbol{w}_h\|_{0,\Omega} &\le \|\boldsymbol{w} - \boldsymbol{w}_h(\boldsymbol{y})\|_{0,\Omega} + \|\boldsymbol{w}_h(\boldsymbol{y}) - \boldsymbol{w}_h\|_{0,\Omega} \\ &\le \|\boldsymbol{w} - \boldsymbol{w}_h(\boldsymbol{y})\|_{0,\Omega} + \|\boldsymbol{y} - \boldsymbol{y}_h\|_{0,\Omega} = \mathcal{O}(h^2). \end{aligned}$$

□

5.3.4 Error bounds in the energy norm

Theorem 5.3.9. *Let $(\boldsymbol{y}, \boldsymbol{w}, p, r)$ be the state and co-state velocities, and pressures, solutions of (5.1)-(5.2), and let $(\boldsymbol{y}_h, \boldsymbol{w}_h, p_h, r_h)$ be their DFV approximations. Then*

$$|||\boldsymbol{y} - \boldsymbol{y}_h|||_{2,h} + \|p - p_h\|_{0,\Omega} = \mathcal{O}(h) \quad \text{and} \quad |||\boldsymbol{w} - \boldsymbol{w}_h|||_{2,h} + \|r - r_h\|_{0,\Omega} = \mathcal{O}(h).$$

Proof. Using (5.32) and (5.33), applying triangle inequality and Lemma 5.3.1, we obtain

$$\begin{aligned} |||\boldsymbol{y} - \boldsymbol{y}_h|||_{2,h} + \|p - p_h\|_{0,\Omega} &\le |||\boldsymbol{y} - \boldsymbol{y}_h(\boldsymbol{u})|||_{2,h} + |||\boldsymbol{y}_h(\boldsymbol{u}) - \boldsymbol{y}_h|||_{2,h} \\ &\quad + \|p - p_h(\boldsymbol{u})\|_{0,\Omega} + \|p_h(\boldsymbol{u}) - p_h\|_{0,\Omega} \\ &\le |||\boldsymbol{y} - \boldsymbol{y}_h(\boldsymbol{u})|||_{2,h} + \|p - p_h(\boldsymbol{u})\|_{0,\Omega} + C \, \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} , \end{aligned}$$

and

$$\begin{aligned}\|\boldsymbol{w}-\boldsymbol{w}_h\|_{2,h}+\|r-r_h\|_{0,\Omega} \le\ & \|\boldsymbol{w}-\boldsymbol{w}_h(\boldsymbol{y})\|_{2,h}+\|\boldsymbol{w}_h(\boldsymbol{y})-\boldsymbol{w}_h\|_{2,h}\\ & +\|r-r_h(\boldsymbol{y})\|_{0,\Omega}+\|r_h(\boldsymbol{y})-r_h\|_{0,\Omega}\\ \le\ & \|\boldsymbol{w}-\boldsymbol{w}_h(\boldsymbol{y})\|_{2,h}+\|r-r_h(\boldsymbol{y})\|_{0,\Omega}+C\,\|\boldsymbol{y}-\boldsymbol{y}_h\|_{0,\Omega}\,.\end{aligned}$$

Therefore, the proof is complete after combining the estimates of Lemma 5.3.2 and the estimates of $\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$ and $\|\boldsymbol{y}-\boldsymbol{y}_h\|_{0,\Omega}$. □

5.4 Numerical experiments

In this Section, we present a set of numerical examples to illustrate the theoretical results previously described. For the sake of completeness, before jumping into the tests we provide some details about the implementation and algorithms for the efficient computation of the DFV method applied to the optimal control of Brinkman equations.

Implementation aspects

We will use the well-known active set strategy (proposed in [8]) involving primal and dual variables (see also [33, 68] for its application in Stokes flow). The principle is to approximate the constrained optimal control problem by a sequence of unconstrained problems, using active sets as summarized in Algorithm 2 below. By $\boldsymbol{u}_h^n, \boldsymbol{w}_h^n$ we will denote the optimal control and adjoint velocity, solutions to the discrete problem (5.23)-(5.27) at the current iteration. Also, the control constraints are $\boldsymbol{u}_a = (u_{a_1}, ..u_{a_d})^T$ and $\boldsymbol{u}_b = (u_{b_1}, ..u_{b_d})^T$.

Let $\{\vec{\phi}_i\}_{i=1}^N$, $\{\xi_i\}_{i=1}^L$, $\{\vec{\psi}_i\}_{i=1}^M$ be the basis functions for $\mathbf{V}_h$, Q_h, and $\mathbf{U}_h$, respectively, whereas the space $\mathbf{V}_h^*$ is spanned by $\{\vec{\phi}_i^*\}_{i=1}^N$, with (explicited here for $d=3$)

$$\vec{\phi}_i^*(\boldsymbol{x}) = \{\chi_{T_i^*}(1,0,0), \chi_{T_i^*}(0,1,0), \chi_{T_i^*}(0,0,1)\},$$

where $\chi_{T_i^*}$ is the characteristic function assuming the value 1 on $T_i^* \in \mathcal{T}_h^*$ and zero elsewhere.

h	$e_0(\boldsymbol{y})$	rate	$e_1(\boldsymbol{y})$	rate	$e_0(p)$	rate	$e_0(\boldsymbol{w})$	rate	$e_1(\boldsymbol{w})$	rate	$e_0(r)$	rate	$e_0(\boldsymbol{u})$	rate	it
Piecewise constant control															
0.7071	0.3025	–	2.2608	–	0.3301	–	0.3025	–	2.2608	–	0.3301	–	0.1824	–	2
0.4714	0.1744	1.3576	1.5583	0.9178	0.3062	0.1856	0.1770	1.3225	1.5621	0.9118	0.3062	0.1856	0.1204	1.0237	3
0.2828	0.0872	1.3570	1.0574	0.7592	0.1995	0.8385	0.0893	1.3381	1.0600	0.8592	0.1995	0.8385	0.0743	0.9452	3
0.1571	0.0316	1.7249	0.6188	0.9114	0.1146	0.9431	0.0326	1.7142	0.6196	0.9133	0.1146	0.9431	0.0416	0.9879	3
0.0832	0.0084	2.0924	0.3327	0.9758	0.0613	0.9829	0.0086	2.0908	0.3328	0.9774	0.0613	0.9829	0.0216	1.0283	3
0.0429	0.0015	2.1599	0.1725	0.9906	0.0317	0.9953	0.0016	2.5745	0.1725	0.9910	0.0317	0.9953	0.0111	1.0031	3
0.0218	0.0004	2.1321	0.0885	0.9838	0.0161	0.9988	0.0004	2.0269	0.0886	0.9827	0.0161	0.9988	0.0053	1.0435	3
0.0110	0.0001	2.0092	0.0464	0.9438	0.0081	0.9997	0.0001	2.0377	0.0465	0.9893	0.0081	0.9997	0.0027	1.0297	3
Piecewise linear control															
0.7071	0.3025	–	2.2608	–	0.3301	–	0.3025	–	2.2608	–	0.3301	–	0.1825	–	2
0.4714	0.1751	1.3479	1.5593	0.9163	0.3062	0.1856	0.1770	1.3222	1.5622	0.9117	0.3062	0.1856	0.0886	1.7827	3
0.2828	0.0876	1.3555	1.0578	0.7596	0.1995	0.8385	0.0893	1.3380	1.0600	0.7592	0.1995	0.8385	0.0540	0.9685	3
0.1571	0.0318	1.7219	0.6190	0.9117	0.1146	0.9431	0.0326	1.7141	0.6196	0.9134	0.1146	0.9431	0.0243	1.3617	3
0.0832	0.0084	2.0898	0.3322	0.9761	0.0613	0.9829	0.0086	2.0906	0.3328	0.9774	0.0613	0.9829	0.0090	1.5563	3
0.0429	0.0015	2.1754	0.1729	0.9907	0.0317	0.9953	0.0016	2.1751	0.1725	0.9910	0.0317	0.9953	0.0032	1.5640	3
0.0218	0.0004	2.1360	0.0882	0.9841	0.0159	0.9991	0.0005	1.9940	0.0887	0.9914	0.0160	0.9994	0.0011	1.5565	2
0.0110	0.0001	2.0107	0.0460	0.9432	0.0075	0.9942	0.0001	2.0278	0.0453	0.9793	0.0079	0.9997	0.0002	1.5718	3

Table 5.1: Example 1: convergence history and optimization iteration count for the approximations of the optimal control of the Brinkman problem.

We next proceed to define the discrete active and inactive sets, based on the degrees of freedom of $\mathbf{U}_h$, as follows

$$\begin{aligned} A_{n+1}^{\boldsymbol{u}_a} &= \{k \in \{1,\dots,M\} : -w_{j,h}^{n,k}/\lambda < u_{a_j}, \text{ for any } j \in \{1,\dots,d\}\}, \\ A_{n+1}^{\boldsymbol{u}_b} &= \{k \in \{1,\dots,M\} : -w_{j,h}^{n,k}/\lambda > u_{b_j}, \text{ for any } j \in \{1,\dots,d\}\}, \\ I_{n+1} &= \{1,\dots,M\} \setminus (A_{n+1}^{\boldsymbol{u}_a} \cup A_{n+1}^{\boldsymbol{u}_b}), \end{aligned} \tag{5.84}$$

where, in general, $s_{j,h}^{n,k}$ stands for the discrete value associated to the degree of freedom at position k, related to the spatial component j of the vector field $\boldsymbol{s}$, at the step n of Algorithm 2. By the definition of the optimal control problem, we have that

$$\boldsymbol{u}_h^n = \begin{cases} \boldsymbol{u}_a & \text{in} \quad A_{n+1}^{\boldsymbol{u}_a}, \\ -\lambda^{-1}\boldsymbol{w}_h^n & \text{in} \quad I_{n+1}, \\ \boldsymbol{u}_b & \text{in} \quad A_{n+1}^{\boldsymbol{u}_b}, \end{cases}$$

and if we further introduce the following characteristic sets

$$\chi_{A^{\boldsymbol{u}_a}_{n+1}(k,k)} = \begin{cases} 1 & \text{if} \quad k \in A^{\boldsymbol{u}_a}_{n+1}, \\ 0 & \text{else}, \end{cases} \qquad \chi_{A^{\boldsymbol{u}_b}_{n+1}(k,k)} = \begin{cases} 1 & \text{if} \quad k \in A^{\boldsymbol{u}_b}_{n+1}, \\ 0 & \text{else}, \end{cases}$$

then we get

$$\lambda^{-1}\boldsymbol{w}_h^n(1 - \chi_{A^{\boldsymbol{u}_a}_{n+1}} - \chi_{A^{\boldsymbol{u}_b}_{n+1}}) + \boldsymbol{u}_h^n = \boldsymbol{u}_a\chi_{A^{\boldsymbol{u}_a}_{n+1}} + \boldsymbol{u}_b\chi_{A^{\boldsymbol{u}_b}_{n+1}}. \tag{5.85}$$

Finally, we define the matrix blocks

$$\begin{aligned}
&\mathbb{A} := [A_h(\vec{\boldsymbol{\phi}}_i, \vec{\boldsymbol{\phi}}_j)]_{1\le i,j\le N},\ \mathbb{C} := [c_h(\vec{\boldsymbol{\phi}}_i, \vec{\boldsymbol{\phi}}_j)]_{1\le i,j\le N},\ \mathbb{B} := [B_h(\xi_i, \vec{\boldsymbol{\phi}}_j)]_{i\le L; j\le N},\\
&\mathbb{M} := [(\vec{\boldsymbol{\phi}}_i, \vec{\boldsymbol{\phi}}^*_j)_{0,\Omega}]_{1\le i,j\le N}, \mathbb{G} := [(\vec{\boldsymbol{\phi}}^*_i, \vec{\boldsymbol{\psi}}_j)_{0,\Omega}]_{i\le N;1 j\le M},\ \mathbb{D} := [(\vec{\boldsymbol{\psi}}_i, \vec{\boldsymbol{\psi}}_j)_{0,\Omega}]_{1\le i,j\le M},\\
&\hat{\mathbb{E}} := \lambda^{-1}(\mathbf{I} - \chi_{A^{\boldsymbol{a}}_{n+1}} - \chi_{A^{\boldsymbol{b}}_{n+1}}),
\end{aligned}$$

along with the vectors

$$\mathrm{F} := [(\boldsymbol{f}, \vec{\boldsymbol{\phi}}^*_i)_{0,\Omega}]_{i\le N},\ \mathrm{Y}_d := [(\boldsymbol{y}_d, \vec{\boldsymbol{\phi}}^*_i)_{0,\Omega}]_{i\le N},\ \hat{\mathrm{S}} := [(\boldsymbol{a}\,\chi_{A^{\boldsymbol{a}}_{n+1}} + \boldsymbol{b}\,\chi_{A^{\boldsymbol{b}}_{n+1}}, \vec{\boldsymbol{\psi}}_i)_{0,\Omega}]_{i\le M},$$

so that after testing (5.85) against $\{\vec{\boldsymbol{\psi}}_i\}_{i=1}^M$ we end up with the following matrix form of the discrete optimal control problem (5.23)-(5.27):

$$\begin{pmatrix} \mathbb{A}+\mathbb{C} & -\mathbb{B}^T & \mathbf{0} & \mathbf{0} & -\mathbb{G} \\ \mathbb{B} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbb{M} & \mathbf{0} & \mathbb{A}+\mathbb{C} & \mathbb{B}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbb{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{\mathbb{E}}\mathbb{G}^T & \mathbf{0} & \mathbb{D} \end{pmatrix} \begin{pmatrix} \mathrm{Y} \\ \mathrm{P} \\ \mathrm{W} \\ \mathrm{R} \\ \mathrm{U} \end{pmatrix} = \begin{pmatrix} \mathrm{F} \\ \mathbf{0} \\ -\mathrm{Y}_d \\ \mathbf{0} \\ \hat{\mathrm{S}} \end{pmatrix}, \tag{5.86}$$

where $\mathrm{Y}, \mathrm{P}, \mathrm{W}, \mathrm{R}$ and U are the coefficients in the expansion of $\boldsymbol{y}_h^{n+1}$, p_h^{n+1}, $\boldsymbol{w}_h^{n+1}$, r_h^{n+1} and $\boldsymbol{u}_h^{n+1}$, respectively, and the hats indicate quantities associated with the previous iteration.

Example 1. We start by assessing the experimental convergence of the proposed scheme applied to the optimal control problem (5.1)-(5.2) defined on the unit square $\Omega = (0,1)^2$. Viscosity, permeability and the weight for the control cost assume the following constant values $\mu = 1$, $\mathbf{K} = \mathbf{I}$, $\lambda = 1$, respectively. The set of admissible controls is

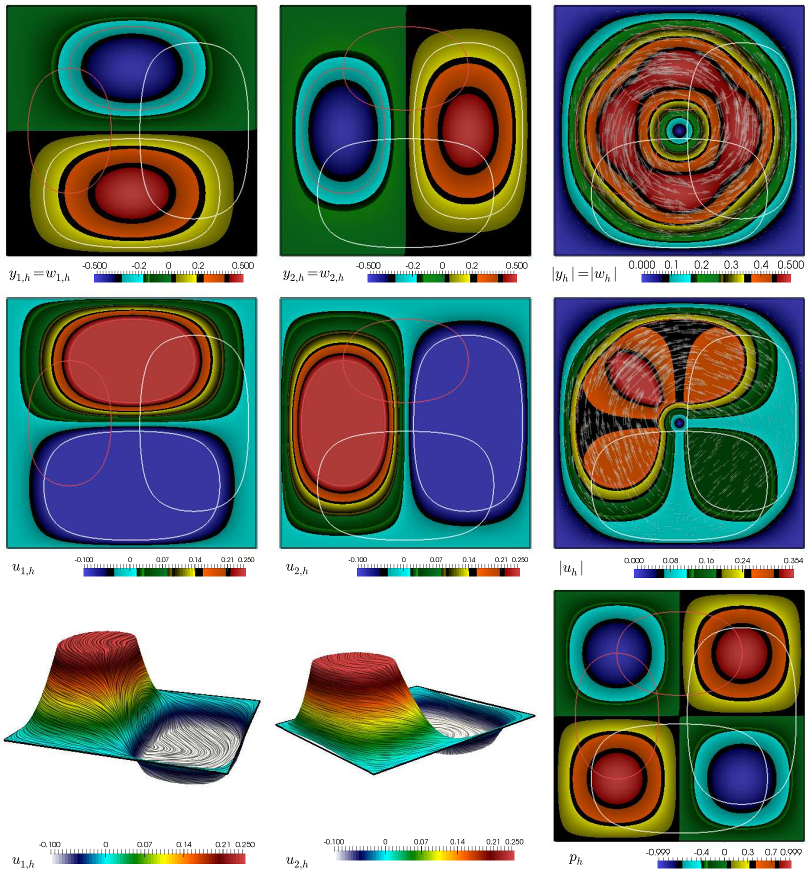


Figure 5.2: Example 1: DFV approximation of state velocity components and magnitude (top panels), components and magnitude of the control variable, here approximated with piecewise linear elements (center row), and state pressure field (bottom row). Contours of the active sets associated to $u_{a_1} = u_{a_2}$ (in white curves) and $u_{b_1} = u_{b_2}$ (red curves) are displayed on each plot.

characterized by the constants $u_{a_1} = u_{a_2} = -\frac{1}{10}$, $u_{b_1} = u_{b_2} = \frac{1}{4}$, and manufactured solutions are explicitly given by

$$\boldsymbol{y} = \boldsymbol{w} = \begin{pmatrix} \sin^2(\pi x_1)\sin(\pi x_2)\cos(\pi x_2) \\ -\sin^2(\pi x_2)\sin(\pi x_1)\cos(\pi x_1) \end{pmatrix}, \; p = -r = \sin(2\pi x_1)\sin(2\pi x_2),$$

$$\boldsymbol{u} = P_{[\boldsymbol{u}_a,\boldsymbol{u}_b]}\left(\frac{-1}{\lambda}\boldsymbol{w}\right),$$

(see e.g. [78]) which satisfy the homogeneous Dirichlet boundary conditions under which the analysis was performed. Source term and desired velocity field of the problem

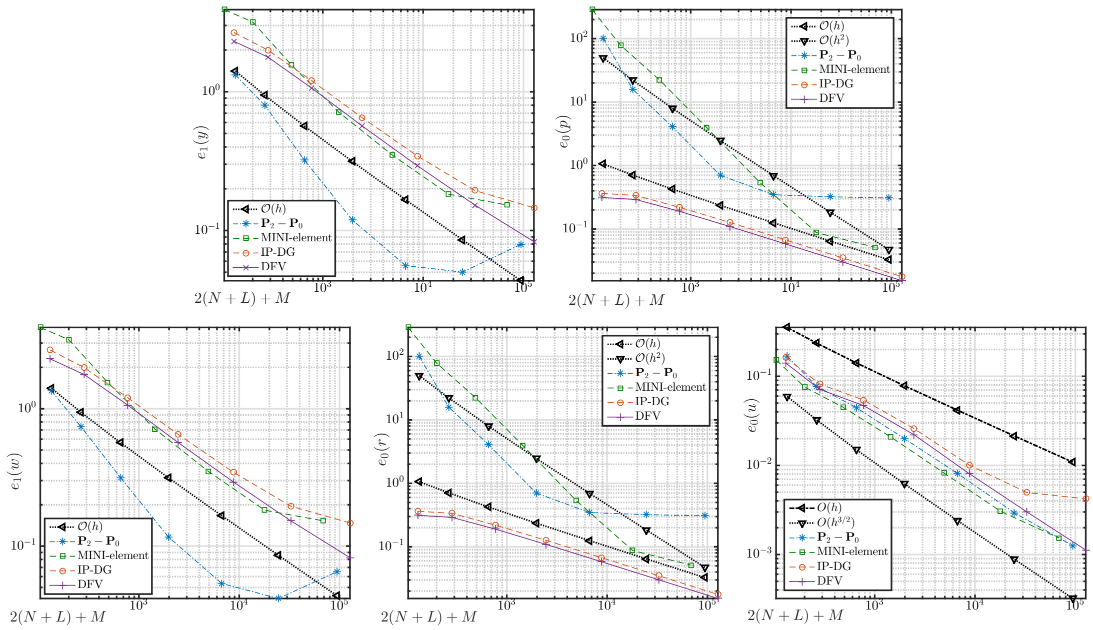


Figure 5.3: Example 1: comparison between errors generated using a $\mathbb{P}_2 - \mathbb{P}_0$, the MINI-element, an interior penalty DG, and a DFV approximation of velocity and pressure in the primal and adjoint problems.

are constructed according to these exact solutions, that is, respectively

$$\boldsymbol{f} = \mathbf{K}^{-1}\boldsymbol{y} - \mathbf{div}(\mu\boldsymbol{\varepsilon}(\boldsymbol{y}) - p\mathbf{I}) - \boldsymbol{u}, \quad \boldsymbol{y}_d = \boldsymbol{y} - \mathbf{K}^{-1}\boldsymbol{w} + \mathbf{div}(\mu\boldsymbol{\varepsilon}(\boldsymbol{w}) + r\mathbf{I}).$$

A family of nested primal and dual triangulations of Ω is generated, on which we compute errors in the $\mathbf{L}^2-$and mesh-dependent norm $\|\!|\cdot|\!\|_{1,h}$ for the state and co-state velocity, in the L^2-norm for pressures, and in the $\mathbf{L}^2-$norm for the control approximation. Table 5.1 displays the error history for this first test, where we observe optimal convergence rates for velocity and pressure (only those of the state equation are shown) in their natural norms, along with an $O(h)$ convergence for the control when approximated by piecewise constant elements, which improves to roughly a $O(h^{3/2})$ rate under piecewise linear approximations. We can also confirm that a maximum of three iterations are needed to reach the stopping criterion that the active sets are equal to those in the previous optimization step. This indicates a mesh independence of the method in the sense that the number of iterations needed to achieve the stopping criterion is independent of the resolution. In addition we portray in Figure 5.2 the obtained approximate solutions at the finest resolution level, where we highlight the active sets with a contour plot on top of the control and state velocities. In all examples herein we employ a BiCGSTAB method with AMG preconditioning to solve the linear systems involved at each step of Algorithm 2. Moreover, the zero-mean pressure condition is applied for both pressure and adjoint pressure using a real Lagrange multiplier approach.

At this point we also present a basic comparison with other classical methods in terms of accuracy. For instance, we have performed the same test as above but employing discontinuous coefficients. Both fluid viscosity and medium permeability have now a jump of five orders of magnitude at $x_1 = 0.5$. The tested methods are: a conforming stable $\mathbb{P}_2 - \mathbb{P}_0$ and MINI-element pairs for velocity and pressure approximation, a classical interior penalty DG method using the same stabilization parameters as in (5.10)-(5.11), and the proposed DFV formulation. In all cases we consider a piecewise linear approximation of the control variable.

The results are collected in Figure 5.3, where convergence histories (errors for velocity and pressure vs. the number of degrees of freedom DoF $= 2(N + L) + M$) associated to the studied discretizations are shown. For all fields, the DFV approximation exhibits a slightly better accuracy than its pure-DG counterpart. This may be explained by the smaller elements used in the dual mesh (but being associated to the same number of DoF). On the other hand, for coarse meshes the conforming approximation $\mathbb{P}_2 - \mathbb{P}_0$ outperforms all other methods, but for finer meshes the discontinuous coefficients of the problem imply a badly conditioned system matrix requiring more (inner) iterations of the linear solver and eventually the conforming methods loose their optimal convergence. For a fixed number of DoF, the proposed DFV scheme produces smaller errors for the pressure approximation than the other methods. We stress that some recent theoretical comparison results are available for forward Stokes problems (see e.g. [15]), but only in the case of smooth solutions and constant coefficients.

Example 2. Our second test focuses on the optimal control problem applied to the well-known lid driven cavity problem. The objective function still corresponds to (5.1), but no analytic exact solution is available. Again the domain consists of the unit square, and the data of the problem are given by a traction boundary condition on the top of the

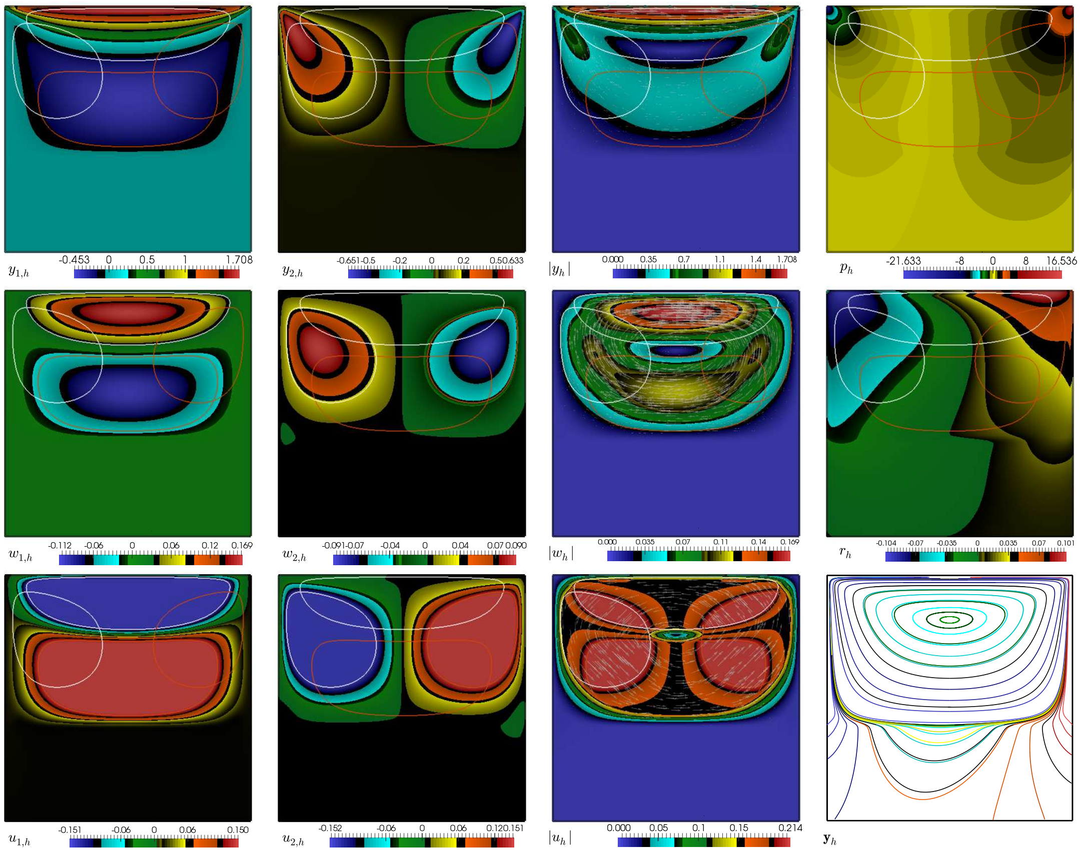


Figure 5.4: Example 2: DFV approximation of state velocity components and magnitude along with state pressure (top panels), adjoint velocity and pressure (center row), components and magnitude of the control variable under piecewise constant approximation, and state velocity streamlines (bottom row). Contours of the active sets associated to $u_{a_1} = u_{a_2}$ (in white curves) and $u_{b_1} = u_{b_2}$ (red curves) are displayed on each plot.

lid, the applied body force, and an observed velocity field defined by:

$$\boldsymbol{y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ on the top and zero elsewhere,} \qquad \boldsymbol{f} = \boldsymbol{y}_d = \boldsymbol{0} \text{ in } \Omega.$$

The adjoint problem is subject to homogeneous Dirichlet data. The viscosity is set to $\mu = 0.1$, the control weight is now $\lambda = 0.2$, the admissible control space is characterized by $u_{a_1} = u_{a_2} = -0.15$, $u_{b_1} = u_{b_2} = 0.15$, and the permeability exhibits a discontinuity on the line $x_2 = 0.4$: $\mathbf{K} = \frac{\kappa}{\mu}\mathbf{I}$, with $\kappa = \{10000 \text{ if } x_2 \geq 0.4; 10 \text{ elsewhere in } \Omega\}$. The domain is discretized into 20000 primal triangular elements, and Figure 5.4

λ	1	0.2	0.04	0.008	0.0016	0.00032	0.000064
it	5	6	7	8	12	19	34

Table 5.2: Example 2: iteration count vs. the regularization parameter for the DFV approximations of the optimal control of the Brinkman problem.

portrays all fields obtained with our DFV scheme, where the stabilization parameter is $\alpha_d = 10$. From Figure 5.4 we observe that the controlled velocity approaches to the desired velocity, that is, it goes to zero and the movement of the fluid concentrates in the upper section of the cavity. In addition, we study the influence of the Tikhonov regularization in the iteration count of the active set algorithm applied to a coarse solve of this test. As in [33], we immediately observe that a larger number of iterations are required for smaller values of λ (see Table 5.2).

Example 3. Next we turn to the numerical solution of a three-dimensional optimal control problem. The domain now consists of a cylinder with height 4 and radius 1, aligned with the x_2 axis. The permeability field is now anisotropic $\mathbf{K} = \mathrm{diag}(0.1, 10^{-6}\chi_B + 0.1\chi_{B^c}, 0.1)$, where B is a ball of radius 1/4 located at the center of the domain. As boundary condition for the state velocity, a Poiseuille inflow profile is imposed at the bottom of the cylinder (i.e. on $x_2 = 0$): $\boldsymbol{y} = (0, 10(1 - x_2^2 - (x_3 - 1/2)^2), 0)^T$, a zero-pressure is considered on $x_2 = 4$, whereas homogeneous Dirichlet data are enforced on the remainder of $\partial\Omega$. The viscosity is constant $\mu = 0.01$, the Tikhonov regularization is $\lambda = 1/2$, the desired velocity is zero $\boldsymbol{y}_d = \mathbf{0}$, the bounds for the control are $u_{a_j} = u_a = -0.1$ and $u_{b_j} = u_b = 0.2$, and a smooth body force is set as in [2]: $\boldsymbol{f} = \mathbf{K}^{-1}(\exp(-x_2x_3) + x_1\exp(-x_2^2), \cos(\pi x_1)\cos(\pi x_3) - x_2\exp(-x_2^2), -x_1x_2x_3 - x_3\exp(-x_3^2))^T$. The primal meshes has 76766 internal tetrahedral elements and 13663 vertices. For this test we observe that five iterations are required to reach the stopping criterion (5.83). Snapshots of the resulting approximate fields are collected in Figure 5.5.

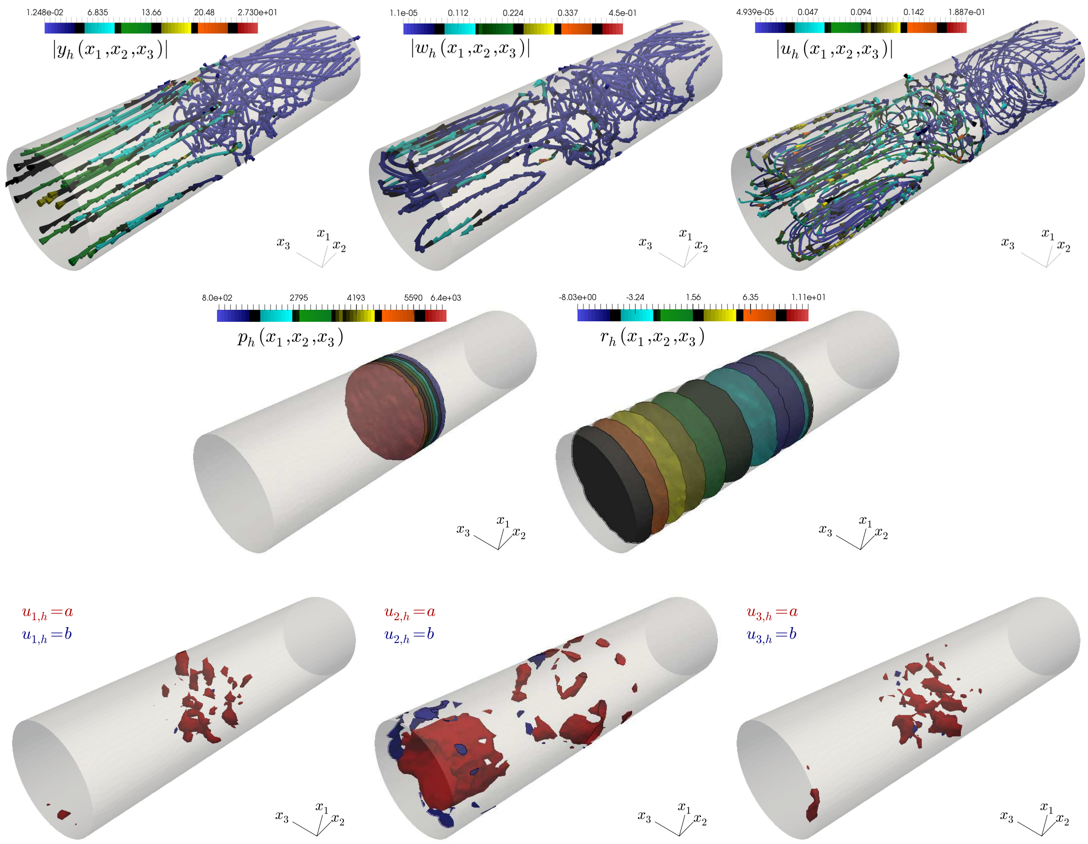


Figure 5.5: Example 3: streamlines of the DFV approximate state and co-state velocities, along with control field (top row), iso-surfaces of approximate state and co-state pressures (middle), and iso-surfaces of the control components associated to $a = u_{a_1} = u_{a_2} = u_{a_3}$ (in red) and $b = u_{b_1} = u_{b_2} = u_{b_3}$ (blue) (bottom panels).

Algorithm 2 Active set implementation and overall solution strategy.

choose **and** store arbitrary initial guess

$$\boldsymbol{u}_h^0 = \boldsymbol{u}_b, \qquad \boldsymbol{w}_h^0 = -\lambda \boldsymbol{u}_h^0$$

initialize active and inactive sets

$$A_0^{\boldsymbol{u}_a} = A_0^{\boldsymbol{u}_b} = \emptyset, \qquad I_0 = \{1, \dots, M\}$$

for $n = 0, 1, \dots,$ **do**

For known $\boldsymbol{u}_h^n$ and $\boldsymbol{w}_h^n$, **construct** the new finite active sets $A_{n+1}^{\boldsymbol{u}_a}$, $A_{n+1}^{\boldsymbol{u}_b}$ as well as the finite inactive set I_{n+1} from (5.84)

if

$$n \geq 1, \;\; \textbf{and} \; A_{n+1}^{\boldsymbol{u}_a} = A_n^{\boldsymbol{u}_a}, \;\; \textbf{and} \; A_{n+1}^{\boldsymbol{u}_b} = A_n^{\boldsymbol{u}_b}, \tag{5.83}$$

then

stop

else

find $\left(\boldsymbol{y}_h^{n+1}, p_h^{n+1}, \boldsymbol{w}_h^{n+1}, r_h^{n+1}, \boldsymbol{u}_h^{n+1}\right)$, solution to the coupled system (5.86)

end if

reinitialize active and inactive sets and control variable

$$A_n^{\boldsymbol{u}_a} \leftarrow A_{n+1}^{\boldsymbol{u}_a}, \quad A_n^{\boldsymbol{u}_b} \leftarrow A_{n+1}^{\boldsymbol{u}_b}, \quad I_n \leftarrow I_{n+1}, \quad \boldsymbol{u}_h^n \leftarrow \boldsymbol{u}_h^{n+1}$$

end for

CHAPTER 6

Concluding Remarks and Future Directions

In this dissertation we have considered DFV methods for the approximation of optimal control problems governed by semilinear elliptic, parabolic and hyperbolic equations, and also by Brinkman equations. The main emphasis was on theoretical and computational aspects of the proposed methods for investigated problems. The main ingredient in the analysis part was the derivation of *a priori* error estimates in suitable norms for the unknown variables that appeared in the formulation. Moreover, several numerical experiments were presented for validation of theoretical error estimates. Now, we summarize the main findings obtained in each Chapter of the thesis and describe the general conclusions based on these findings. Furthermore, we present possible extensions of this work.

6.1 Summary

Chapter 1 dealt with review and applications of optimal control problems governed by a class of PDEs. In this Chapter, we have clearly mentioned the suitability and advantages of the proposed method in comparison with the other existing numerical schemes such as FE, FV and DG methods. Here, two different strategies: *optimize-then-discretize* and *discretize-then-optimize*–generally, used for solvability of optimal control problems was discussed, and justification of employing *optimize-then-discretize* approach was mentioned.

In Chapter 2, we have studied DFV approximations for semilinear elliptic optimal control problems. In this Chapter, first we have considered linear elliptic problem, because of the following reasons. First, these kind of problems have numerous applications and second, there are contributions which dealt with DFV approximations for linear elliptic problems. Also, the analysis presented for this case can be easily extended to semilinear and Brinkman optimal control problems-which are the problems of our interest. In this Chapter, by following the analysis of [49], we have established optimal *a priori* error estimates for DFV methods applied to linear and semilinear elliptic optimal control problems. Moreover, for numerical solution of nonlinear algebraic equations–obtained after DFV discretization of semilinear elliptic equation, an interpolated coefficient method was employed. It has been shown that this idea has computational advantages (in terms of computation of the jacobian) compared with the standard Newton method.

Chapter 3 is devoted to the development of DFV methods for semilinear parabolic optimal control problems. Both semidiscrete and fully discrete scheme were discussed in detail, and existence of a unique local optimal control was examined. Error analysis for both the schemes in mesh and time dependent norms has been carried out for all three discretization of control variables. Numerical examples were presented by considering the applications of these problems related to controlled heating of a body and to justify the theoretical findings. In order to solve the resulting nonlinear system of equations, the idea of interpolated coefficient was exploited.

Since the treatment of hyperbolic problems is similar to the parabolic problems except the time derivative, in Chapter 4, we have extended the analysis of Chapter 3 to the semilinear hyperbolic optimal control problems. The time derivative was approximated by applying an implicit difference scheme, and for discretization of space, linear DFV methods was used. Error estimates for semi-discrete scheme are derived which were analogous to the estimates for parabolic case. In order to demonstrate the real life applications of these problems, in our numerical experiments, a membrane problem is considered.

Considering the applications of fluid control problems and FV methods in computational fluid dynamics, Chapter 5 is dedicated to describe the DFV approximations of optimal control problems governed by Brinkman equations. By following the analysis of [52] which dealt with DFV approximations of Stokes equations, a detailed error analysis has been carried out. Numerical examples consisting lid driven cavity problem and a cylindrical flow were presented in order to illustrate the performance of the proposed method and validate the predicted rate of convergence. Moreover, through our numerical experiments, a comparison study with other existing classical schemes in terms of accuracy and efficiency was made.

6.2 Concluding Remarks

We would like to make the following comments on theoretical and computational aspects of DFV approximations applied to optimal control problems listed in Chapters 2 to 5.

- The derivation of the optimal error estimate of $\mathcal{O}(h^2)$ for state and costate variables in the L^2- norm with variational discretization of control, was straightforward and was achieved by decomposing the errors and using the estimate of control. However, using the similar arguments for the case of piecewise linear and piecewise constant discretizations of control leads to suboptimal order of convergence for state and costate variables. In order to obtain the optimal error estimates, we have used the duality arguments.

- In the case of variational discretization, we were able to derive $\mathcal{O}(h^2)$ convergence rate for control by following the standard arguments. But, we could obtain only $\mathcal{O}(h^{3/2})$ and $\mathcal{O}(h)$ convergence order for control when piecewise linear and constant discretization techniques are used, respectively. Hence, theoretically this approach has advantages over the others. However, there would be some computational difficulties with this scheme (variational discretization), and this can be explained as follows. Since here the control set is not discretized explicitly but discretized by a projection of the discrete costate variables, we observed that the discrete control does not belong to the finite dimensional space associated with mesh and hence one would need to handle nonstandard numerical algorithms and require some advanced tools in order to set the stopping criteria.

- We would also like to mention in order to resolve nonlinear problems, Newton method is employed in general. This requires the computation of Jacobian matrix (which involves derivative) at each iteration which is very time consuming and expensive. For tackling these difficulties, we have utilized the idea of interpolated coefficients together with DFV methods to approximate semilinear elliptic, parabolic and hyperbolic optimal control problems. It was observed that with the introduction of interpolated coefficients the computation was cheap and the Jacobian matrix was computed in a simple way, as the derivative of nonlinear term involved direct multiplication with mass matrix and Jacobian matrix was updated in each iteration of the Newton method.

- Computationally DFV methods would be advantageous compared to classical FE, FV and DG methods. This is because the size of the control volume in DFV methods is almost half of the control volume used in continuous FV methods and test space is piecewise constant (same as in continuous FV). Moreover, this proposed method enjoy desirable features of both DG and FV methods.

6.3 Future Directions

Considering the applications of DFV methods, in immediate future, we will concentrate on the DFV approximations of the optimal control problems subject to partial differential equations governing flow-based phenomena. We aim at the development of specialized solution techniques and mathematical analysis that can allow us to put into proper perspective the framework for studying these processes. On the lines of our work presented in this thesis, we will exploit the essential advantages of both FVM and DG methods, now in the context of more application based optimal control problems. We outline a few milestones as follows:

6.3.1 DFV methods for convection-dominated diffusion optimal control problems

Optimal control for convection-diffusion equation is widely met in real life applications such as the shape optimization of technological devices, the identification of parameters in environmental processes and flow control problems. In environmental science, some phenomena modeled by linear convection-diffusion partial differential equations are often studied to investigate the distribution forecast of pollutants in water or in atmosphere. In this context, it might be of interest to regulate the source term of the convection-diffusion equation so that the solution is as near as possible to a desired one, e.g. to operate the emission rates of industrial plants to keep the concentration of pollutants near (or below) a desired level. This problem can be conveniently accommodated in the optimal control framework for convection diffusion equation. For our future study, we will consider the following distributed optimal control problem governed by the unsteady time dependent diffusion-convection reaction equation with control constraints

$$\min_{u\in U_{ad}} J(y,u) := \frac{1}{2}\int_0^T \left(\|y-y_d\|^2_{0,\Omega} + \frac{\lambda}{2}\|u\|^2_{0,\Omega} \right) dt,$$

subject to

$$\left.\begin{aligned} \partial_t y - \varepsilon\Delta y + \vec{b}\cdot\nabla y + ay = \mathcal{B}u + f \quad \text{in} \quad (0,T)\times\Omega, \\ y(t,x) = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \\ y(x,0) = y_0(x), \quad x\in\Omega. \end{aligned}\right\}$$

with the set of admissible controls defined by

$$U_{ad} = \{u \in \mathcal{U} = L^2(L^2) : u_a \leqslant u(x) \leqslant u_b,\ a.e. \text{ in } \Omega\},$$

with bounds $u_a, u_b \in \mathbb{R}$ that fulfill $u_a < u_b$. The domain Ω is bounded, open and convex in $\mathbb{R}^d$, $d = 2,3$, with Lipschitz boundary $\partial\Omega$. The source function f and the desired state $y_d \in L^2(L^2)$. The initial condition $y_0(x) \in H^1_0(\Omega)$. Here, $a > 0$ is the reaction coefficient, $0 < \varepsilon << 1$ is a small positive number. The given velocity field $\vec{b} \in W^{1,\infty}(\Omega)^2$ satisfies the incompressibility condition, i.e $\nabla\cdot\vec{b} = 0$. We also assume that the following coercivity condition holds:

$$a - \frac{1}{2}\nabla\cdot\vec{b} \geq 0 > 0.$$

For this problem, we will focus on the following:

- Development of suitable DFV schemes.
- Convergence analysis of the proposed scheme.
- Efficient implementation and numerical solution of problems with application-based interest.

6.3.2 DFV approximations for the optimal control problems governed by coupled flow-transport equations

We expect the advantages of DFV methods to be much more evident in presence of more complicated domain heterogeneities, high solution gradients, nonlinearities, and coupling with other transport phenomena modelling, e.g. thermal effects or sedimentation-consolidation of small particles within viscous fluids. In view of these, we consider the following optimal control problem:

$$\min_{u \in U_{ad}} J(u) := \frac{1}{2}\|\phi - \phi_d\|_{0,\Omega}^2 + \frac{\lambda}{2}\|u\|_{0,\Omega}^2,$$

subject to the following transport equation together with Stokes problem:

$$\begin{aligned}
\partial_t \phi - \operatorname{div}(\kappa(\phi)\nabla\phi) + \boldsymbol{y} \cdot \nabla\phi = \mathcal{B}u + \nabla \cdot \boldsymbol{f}(\phi) \quad &\text{in } (0,T) \times \Omega,\\
-\mathbf{div}\big(\mu(\phi)\boldsymbol{\varepsilon}(\boldsymbol{y}) - p\mathbf{I}\big) - \phi\boldsymbol{g} = \mathbf{0} \quad &\text{in } (0,T) \times \Omega,\\
\operatorname{div} \boldsymbol{y} = 0 \quad &\text{in } (0,T) \times \Omega,\\
\boldsymbol{y} = 0 \quad &\text{on } (0,T) \times \partial\Omega,\\
\phi = 0 \quad &\text{on } (0,T) \times \partial\Omega,\\
\phi(0) = \phi_0 \quad &\text{on } \{0\} \times \Omega.
\end{aligned}$$

Here, the desired concentration field ϕ_d is assumed to be from $C^{0,\sigma}(\bar{\Omega})^d$, $\sigma \in (0,1)$ and

$$U_{ad} := \{u(t,x) \in U = L^\infty(L^\infty) : \ a \le u(t,x) \le b, \ \ a.e.\,(t,x) \in (0,T) \times \Omega; \ a, b \in \mathbb{R}\}.$$

The above model problem describe the motion of an incompressible mixture and the evolution of the solids concentration. The primal unknowns are the volume average flow velocity of the mixture $\boldsymbol{y}$, the solids concentration ϕ, and the pressure field p and u which is the control variable. In addition, $\mu(\phi)\boldsymbol{\varepsilon}(\boldsymbol{y}) - p\boldsymbol{I}$ is the Cauchy stress tensor, $\boldsymbol{\varepsilon}(\boldsymbol{y}) = \frac{1}{2}(\nabla\boldsymbol{y} + \nabla\boldsymbol{y}^{\mathrm{T}})$ is the infinitesimal rate of strain, and $\mu = \mu(\phi)$ is the concentration-dependent viscosity.

For the numerical approximation, we will pay close attention to the following:

- Development and a priori error analysis of DFV schemes for above mentioned problem.

- Derivation of *a priori* error estimates.
- Implementation of mesh adaptivity and validation using benchmark solutions.
- To investigate the exploitability of smart preconditioners and efficient solvers.

6.3.3 DFV methods for optimal control of Brinkman flows with pressure based optimality conditions

In future, we would also like to investigate about the optimal control with different formulations for Brinkman flows (including e.g. vorticity-based systems), along with unconstrained and pressure-based optimality conditions. In this direction, we wil consider DFV approximations for the following optimal control problem:

$$\min_{\boldsymbol{y},p,\boldsymbol{u}} \frac{1}{2}\|\boldsymbol{y}-\boldsymbol{y}_d\|^2_{0,\Omega} + \frac{\delta}{2}\|p-p_d\|^2_{0,\Omega} + \frac{\lambda}{2}\|\boldsymbol{u}\|^2_{0,\Omega},$$

governed by the Brinkman equations

$$\left.\begin{aligned} \mathbf{K}^{-1}(\boldsymbol{x})\boldsymbol{y} - \mathbf{div}(\mu(\boldsymbol{x})\boldsymbol{\varepsilon}(\boldsymbol{y}) - p\mathbf{I}) = \boldsymbol{u} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{y} = 0 \quad \text{in } \Omega, \\ \boldsymbol{y} = \boldsymbol{w} \quad \text{on } \partial\Omega. \end{aligned}\right\}$$

Here, $\boldsymbol{u}$ denotes the forcing term on the right hand side, which is known as control. $\lambda > 0$ is the Tikohonov regularization parameter, $\delta > 0$ is a constant added in front of the desired pressure to enable us to penalize the pressure. The idea is to choose the forcing term $\boldsymbol{u}$ such that the velocity $\boldsymbol{y}$ and pressure p are as close as possible to $\boldsymbol{y}_d$ and p_d in some sense, while still satisfying the Brinkman equations.

So far we have considered only distributed optimal control problems. As a part of our future work we are also interested in the investigation of more applied control setting by considering boundary control problems.

6.3.4 DFV methods for optimal Dirichlet boundary control for the Navier-Stokes equations

We would like to extend DFV approximations for the Dirichlet boundary control problem governed by Navier-Stokes equations. We will investigate the following velocity tracking problem:

$$\min_{\boldsymbol{y},\boldsymbol{u}} \frac{1}{2}\int\limits_{\Omega} |\boldsymbol{y}-\boldsymbol{y}_d|^2 \,\mathrm{d}\boldsymbol{x} + \frac{\lambda}{2}\int\limits_{\partial\Omega} |\boldsymbol{u}|^2 \,\mathrm{d}s,$$

governed by Navier-Stokes equations

$$\left.\begin{aligned} -\Delta\boldsymbol{y} + (\boldsymbol{y}\cdot\nabla)\boldsymbol{y} + \nabla p = \boldsymbol{f} &\quad \text{in } \Omega,\\ \nabla\cdot\boldsymbol{y} = 0 &\quad \text{in } \Omega,\\ \boldsymbol{y} = \boldsymbol{u} &\quad \text{on } \partial\Omega. \end{aligned}\right\}$$

The main idea of the model problem is to influence and eventually drive the velocity vector field $\boldsymbol{y}$ to a given target field $\boldsymbol{y}_d$, by using a control function $\boldsymbol{u}$ on the boundary of the domain Ω.

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PUBLICATIONS BASED ON THESIS

Referred Journal Publications

1. Ruchi Sandilya and Sarvesh Kumar, "Convergence analysis of discontinuous finite volume methods for elliptic optimal control problems", *International Journal of Computational Methods*, Vol. 13, No.2, 1640012 (20 pages), 2016.

2. Ruchi Sandilya and Sarvesh Kumar, "On discontinuous finite volume approximations for semilinear parabolic optimal control problems", *International Journal of Numerical Analysis and Modeling*, Vol. 13, No. 4, Pages 545–568, 2016.

3. Ruchi Sandilya and Sarvesh Kumar, "Convergence of discontinuous finite volume discretizations for a semilinear hyperbolic optimal control problem", *International Journal of Numerical Analysis and Modeling*, Vol. 13, No. 6, Pages 926–950, 2016.

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2. Ruchi Sandilya and Sarvesh Kumar, "A discontinuous interpolated finite volume approximation of semilinear elliptic optimal control problems", *Numerical Methods for Partial Differential Equations*.

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1. R. Sandilya and S. Kumar , "Discontinuous Galerkin finite volume element methods for elliptic optimal control problems", *Online proceedings of ICCM*, Cambridge, England, Vol. 1, ISSN 2374–3948, 2014.

2. R. Sandilya and S. Kumar , "Discontinuous finite volume methods for parabolic optimal control problems", *Mathematical Sciences International Research Journal*, Vol. 4(2), ISSN 2278-8697, 15–22, 2015.